DP Analysis & Approaches (SL/HL)

Maths with Lemon

Hard Probability Questions

Question 1 **[Maximum mark: 11]**

Mia and Oliver play a game with a box containing two green marbles and three blue marbles. Each player in turn randomly selects a marble from the box, notes its colour and replaces it. Mia wins the game if she selects a green marble and Oliver wins the game if he selects a blue marble. Mia starts the game.
Find the probability that

(a) Mia wins on her first turn; [1]

(b) Mia wins on her second turn; [2]

(c) Oliver wins on one of his first three turns; [4]

(d) Oliver eventually wins. [4]

Question 2 **[Maximum mark: 15]**

Bag A contains a large number of tokens. Each token is numbered 1, 2, 3 or 4. A token is selected from the bag and the following table shows the probability distribution of the score, X, obtained.

x	1	2	3	4
$P(X = x)$	p	p	$0.3 - p^2$	$0.4 - 0.7p$

(a) Find the value of p. [2]

(b) Hence, find the value of $E(X)$. [2]

Bag B also contains a large number of tokens, each numbered 1, 2, 3 or 4. A token is selected from the bag and the following table shows the probability distribution of the score, Y, obtained.

y	1	2	3	4
$P(Y = y)$	q	q	s	s

(c) State the range of possible values of s. [3]

(d) Hence, find the range of possible values of $E(Y)$. [3]

A token is selected simultaneously from Bag A and Bag B. The probability of obtaining tokens showing the same score is equal to 0.24.

(e) Calculate $E(Y)$. [5]

Question 3 **[Maximum mark: 11]**

Sophie and Ella play a game. They each have five cards showing Roman numerals I, V, X, L, C. Sophie lays her cards face up on the table in order I, V, X, L, C.
Ella shuffles her cards and lays them face down on the table. She then turns them over one by one to see if her card matches with Sophie's card directly above. Sophie wins if no matches occur; otherwise Ella wins.

(a) Show that the probability that Sophie wins the game is

$$\frac{11}{30}.$$

[6]

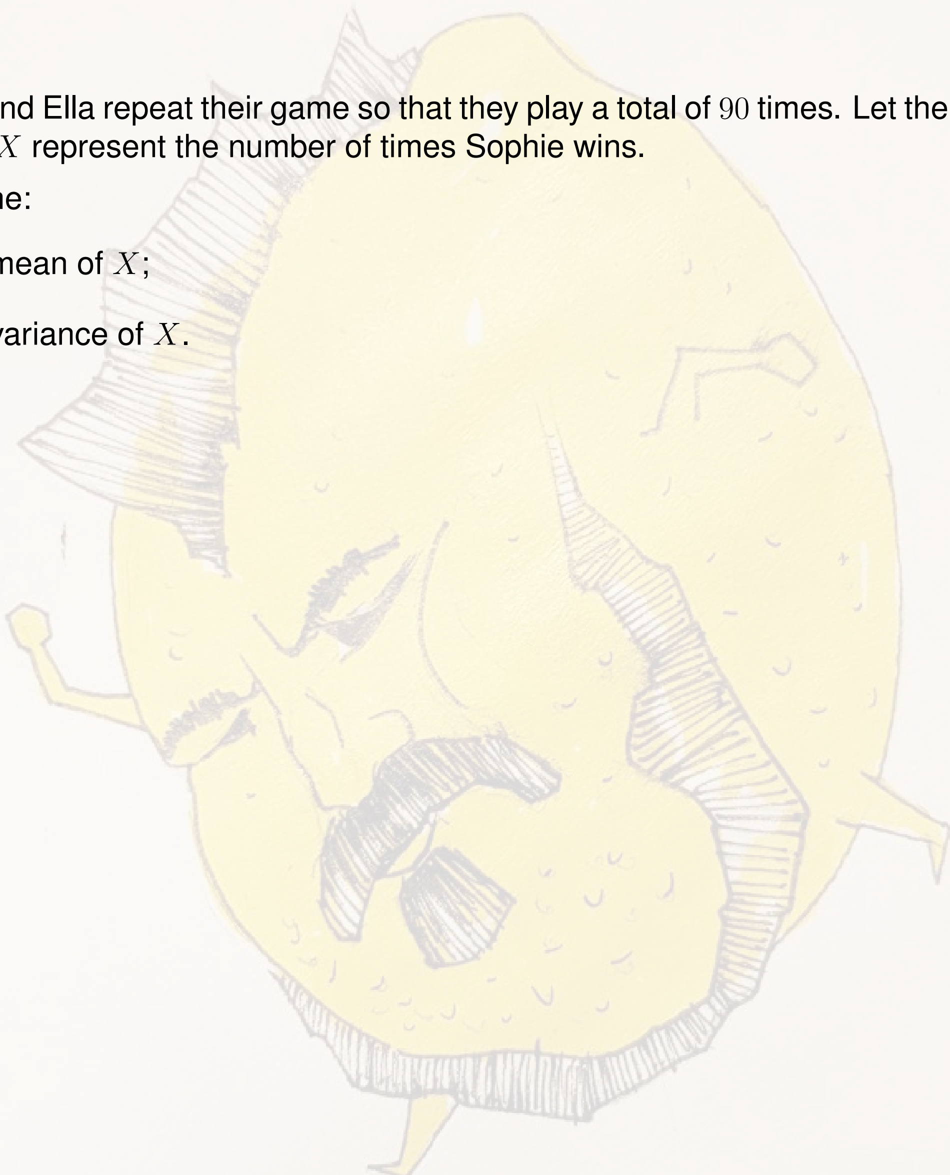
(b) Sophie and Ella repeat their game so that they play a total of 90 times. Let the discrete random variable X represent the number of times Sophie wins.

Determine:

(i) the mean of X;

(ii) the variance of X.

[5]



Question 4 **[Maximum mark: 21]**

Ten balls numbered

$$2, 2, 3, 3, 3, 5, 5, 5, 5, 5$$

are placed in a bag. Balls are taken one at a time from the bag at random and the number noted. Throughout the question a ball is always replaced before the new ball is taken.

(a) A single ball is taken from the bag. Let X denote the value shown on the ball. Find $E[X]$. [2]

(b) Three balls are taken from the bag. Find the probability that:

(i) the sum of the three numbers is 9;

(ii) the median of the three numbers is 3.

[6]

(c) Twelve balls are taken from the bag. Find the probability that more than five of the balls are numbered 2. [3]

(d) Determine the least number of balls that must be taken from the bag for the probability of taking out at least one ball numbered 5 to be greater than 0.9. [3]

(e) Another bag also contains balls numbered 2, 3 or 5. Nine balls are to be taken from this bag at random. It is calculated that the expected number of balls numbered 2 is 4.5, and the variance of the number of balls numbered 3 is 1.25.

Find the least possible number of balls numbered 5 in this bag. [7]

Question 5 **[Maximum mark: 22]**

Ten balls numbered

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4$$

are placed in a bag. Balls are taken one at a time from the bag at random and the number noted. Throughout the question a ball is always replaced before the new ball is taken.

(a) A single ball is taken from the bag. Let X denote the value shown on the ball. Find $E[X]$. [2]

(b) Three balls are taken from the bag. Find the probability that:

 (i) the sum of the three numbers is 8;

 (ii) the median of the three numbers is 4.

[6]

(c) Fifteen balls are taken from the bag. Find the probability that less than six of the balls are numbered 3. [3]

(d) Find the least number of balls that must be taken from the bag for the probability of taking out at least one ball numbered 4 to be greater than 0.95. [3]

(e) Another bag also contains balls numbered 1, 2, 3 or 4. Twenty balls are to be taken from this bag at random. It is calculated that the expected number of balls numbered 2 is 2.5, the expected number of balls numbered 3 is 5, and the variance of the number of balls numbered 1 is 1.8.

Find the least possible number of balls numbered 4 in this bag. [8]

Question 6 **[Maximum mark: 25]**

Jack and John have decided to play a game. They will be rolling a die seven times. One roll of a die is considered as one round of the game. On each round, John agrees to pay Jack \$4 if 1 or 2 is rolled. Jack agrees to pay John \$2 if 3, 4, 5 or 6 is rolled, and who is paid wins the round. In the end, who earns money wins the game.

(a) Show that the probability that Jack wins exactly two rounds is

$$\frac{224}{729}.$$

[3]

(b) (i) Explain why the total number of outcomes for the results of the seven rounds is 128.

(ii) Expand $(1+y)^7$ and choose a suitable value of y to prove that

$$128 = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}.$$

(iii) Give a meaning of the equality above in the context of the seven rounds.

[4]

(c) (i) Find the expected amount of money earned by each player in the game.

(ii) Who is expected to win the game?

(iii) Is this game fair? Justify your answer.

[3]

(d) Jack and John decide to play the game again.

(i) Find an expression for the probability that John wins five rounds on the first game and two rounds on the second game. Give your answer in the form

$$\binom{7}{r}^2 \left(\frac{1}{3}\right)^s \left(\frac{2}{3}\right)^t,$$

where r, s, t are to be found.

(ii) Use your answer to part (d)(i) and seven similar expressions to write down the probability that John wins a total of seven rounds over two games as the sum of eight probabilities.

(iii) Hence prove that

$$\binom{14}{7} = \binom{7}{0}^2 + \binom{7}{1}^2 + \binom{7}{2}^2 + \binom{7}{3}^2 + \binom{7}{4}^2 + \binom{7}{5}^2 + \binom{7}{6}^2 + \binom{7}{7}^2.$$

[9]

(e) Jack and John now roll the die 12 times. Let A denote the number of rounds Jack wins. The expected value of A can be written as

$$E[A] = \sum_{r=0}^{12} r \binom{12}{r} \left[\frac{a^{12-r}}{b^{12}}\right].$$

(i) Find the value of a and b.

(ii) Differentiate the expansion of $(1+y)^{12}$ to prove that the expected number of rolls Jack wins is 4.

[6]

Question 7 **[Maximum mark: 20]**

An ice cream company advertises free gifts to customers who collect four coupons. The coupons are placed at random into 20% of all bags of ice cream sticks. Oliver buys ice cream sticks and he opens them one at a time to see if they contain a coupon.

Let $P(X = n)$ be the probability that Oliver obtains his fourth coupon on the nth ice cream stick opened.

It is assumed that the probability that an ice cream stick contains a coupon stays at 20% throughout the question.

(a) Show that

$$P(X = 4) = 0.0016 \qquad \text{and} \qquad P(X = 5) = 0.00512.$$

[3]

(b) It is given that

$$P(X = n) = \frac{1}{3750}(n-1)(n^2 + an + b)(0.8)^{n-4}$$

for $n \geq 4$, $n \in \mathbb{N}$, where $a, b \in \mathbb{Z}$.

Find the values of a and b. [5]

(c) Deduce that

$$\frac{P(X = n)}{P(X = n-1)} = \frac{4(n-1)}{5(n-4)} \qquad \text{for } n > 4.$$

[4]

(d) (i) Hence show that X has two modes m_1 and m_2.

(ii) Write down the values of m_1 and m_2.

[5]

(e) Oliver's mother goes to the market and buys N ice cream sticks. She takes the ice cream sticks home for Oliver to open.

Determine the minimum value of N such that the probability Oliver receives at least one free gift is greater than 0.5. [3]

Maths with Lemon

Hard Probability Practice