DP Analysis & Approaches (SL/HL)

Maths with Lemon

Hard Probability Questions — Markscheme

Question 1

(a)

$$P(\text{Mia wins on first turn}) = \frac{2}{5} = \boxed{0.4}.$$

(b) For Mia to win on her second turn: Mia first gets blue, Oliver gets green, then Mia gets green.

$$P = \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} = \boxed{\frac{12}{125}} = \boxed{0.096}.$$

(c) Oliver can win on his first, second or third turn:

$$P = \left(\frac{3}{5}\right)^2 + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^3 + \left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^4.$$

$$\boxed{P \approx 0.467}.$$

(d) Oliver eventually wins with probability

$$\left(\frac{3}{5}\right)^2 + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^3 + \left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^4 + \cdots$$

This is geometric with

$$u_1 = \frac{9}{25}, \qquad r = \frac{6}{25}.$$

Therefore

$$P = \frac{9/25}{1 - 6/25} = \frac{9}{19} \approx \boxed{0.474}.$$

Question 2

(a)

$$p + p + 0.3 - p^2 + 0.4 - 0.7p = 1.$$

Hence

$$-p^2 + 1.3p - 0.3 = 0,$$

or

$$10p^2 - 13p + 3 = 0.$$

Thus

$$(10p - 3)(p - 1) = 0.$$

Since $p < 1$,

$$\boxed{p = 0.3}.$$

(b) The distribution is

x	1	2	3	4
$P(X = x)$	0.3	0.3	0.21	0.19

so

$$E(X) = 1(0.3) + 2(0.3) + 3(0.21) + 4(0.19) = \boxed{2.29}.$$

(c)

$$q + q + s + s = 1 \quad \Rightarrow \quad q + s = \frac{1}{2}.$$

Since $q > 0$ and $s > 0$,

$$\boxed{0 < s < \frac{1}{2}}.$$

(d)

$$E(Y) = q + 2q + 3s + 4s = 3q + 7s.$$

Since $q + s = \frac{1}{2}$,

$$E(Y) = 3\left(\frac{1}{2} - s\right) + 7s = \frac{3}{2} + 4s.$$

Thus

$$\boxed{\frac{3}{2} < E(Y) < \frac{7}{2}}.$$

(e) Equal scores have probability

$$0.30q + 0.30q + 0.21s + 0.19s = 0.24.$$

Thus

$$0.6q + 0.4s = 0.24.$$

Together with

$$q + s = \frac{1}{2},$$

we obtain

$$q = 0.2, \qquad s = 0.3.$$

Hence

$$E(Y) = \frac{3}{2} + 4(0.3) = \boxed{2.7}.$$

Question 3

(a) There are

$$5! = 120$$

possible arrangements.
Count arrangements with at least one match:

exact number of matches	number counted in the source solution
1	$\binom{5}{1}(9) = 45$
2	$\binom{5}{2}(2) = 20$
3	$\binom{5}{3}(1) = 10$
4	$\binom{5}{4}(0) = 0$
5	$\binom{5}{5}(1) = 1$

The source markscheme obtains

$$120 - (45 + 20 + 10 + 0 + 1) = 44$$

arrangements with no matches. Therefore

$$P(\text{Sophie wins}) = \frac{44}{120} = \boxed{\frac{11}{30}}.$$

(b)

$$X \sim B\left(90, \frac{11}{30}\right).$$

(i)

$$E(X) = np = 90 \cdot \frac{11}{30} = \boxed{33}.$$

(ii)

$$\mathrm{Var}(X) = np(1-p) = 90 \cdot \frac{11}{30}\left(1 - \frac{11}{30}\right) = \boxed{20.9}.$$

Question 4

(a)

$$P(X=2)=\frac{2}{10},\quad P(X=3)=\frac{3}{10},\quad P(X=5)=\frac{5}{10}.$$

Hence

$$E(X)=2\left(\frac{2}{10}\right)+3\left(\frac{3}{10}\right)+5\left(\frac{5}{10}\right)=\boxed{3.8}.$$

(b)(i) For a sum of 9:

$$P=3P(225)+P(333)$$

$$=3\left(\frac{2}{10}\frac{2}{10}\frac{5}{10}\right)+\left(\frac{3}{10}\right)^3=\boxed{0.087}.$$

(b)(ii) For median 3:

$$P=3P(233)+P(333)+3P(335)+6P(235).$$

Therefore

$$\boxed{P=0.396}.$$

(c) Let Y be the number of 2s among 12 draws.

$$Y\sim B(12,0.2).$$

Thus

$$P(Y>5)=\boxed{0.0194}\quad \text{(approximately)}.$$

(d) Let Z be the number of 5s in N draws.

$$Z\sim B(N,0.5).$$

We require

$$P(Z\geq 1)>0.9.$$

So

$$1-P(Z=0)>0.9$$

and hence

$$(0.5)^N<0.1.$$

The least integer is

$$\boxed{N=4}.$$

(e) Let S be the number of 2s and T the number of 3s in 9 draws.

$$S\sim B(9,p),\qquad T\sim B(9,q).$$

From

$$E(S)=9p=4.5,$$
$$p=\frac{1}{2}.$$

Also

$$\mathrm{Var}(T)=9q(1-q)=1.25,$$

giving

$$q=\frac{1}{6}\quad\text{or}\quad\frac{5}{6}.$$

Since

$$\frac{1}{2}+\frac{5}{6}>1,$$

we must have

$$q = \frac{1}{6}.$$

Therefore

$$P(5) = 1 - \frac{1}{2} - \frac{1}{6} = \frac{1}{3} = \frac{2}{6}.$$

The least possible total number of balls is 6, so the least possible number of 5s is

$$\boxed{2}.$$

Question 5

(a)

$$E(X) = 1\left(\frac{1}{10}\right) + 2\left(\frac{2}{10}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{4}{10}\right) = \boxed{3}.$$

(b)(i)

$$P(\text{sum} = 8) = 6P(134) + 3P(233) + 3P(224).$$

Thus

$$P = 6\left(\frac{1}{10}\frac{3}{10}\frac{4}{10}\right) + 3\left(\frac{2}{10}\frac{3}{10}\frac{3}{10}\right) + 3\left(\frac{2}{10}\frac{2}{10}\frac{4}{10}\right) = \boxed{0.174}.$$

(b)(ii)

$$P(\text{median} = 4) = 3P(144) + 3P(244) + 3P(344) + P(444).$$

Hence

$$\boxed{P = 0.352}.$$

(c) Let Y be the number of 3s in 15 draws.

$$Y \sim B(15, 0.3).$$

Therefore

$$P(Y < 6) = \boxed{0.722} \quad \text{(approximately)}.$$

(d) Let Z be the number of 4s in N draws.

$$Z \sim B(N, 0.4).$$

Require

$$P(Z \geq 1) > 0.95,$$

so

$$(0.6)^N < 0.05.$$

The least integer is

$$\boxed{N = 6}.$$

(e) Let probabilities for 1, 2 and 3 be p, q, r respectively.
From

$$E(T) = 20q = 2.5,$$

$$q = \frac{1}{8}.$$

From

$$E(U) = 20r = 5,$$

$$r = \frac{1}{4}.$$

From

$$20p(1-p) = 1.8,$$

$$p = \frac{1}{10} \quad \text{or} \quad \frac{9}{10}.$$

Since

$$\frac{9}{10} + \frac{1}{8} + \frac{1}{4} > 1,$$

we take

$$p = \frac{1}{10}.$$

Therefore

$$P(4) = 1 - \left(\frac{1}{10} + \frac{1}{8} + \frac{1}{4} \right) = \frac{21}{40}.$$

The minimum total number of balls is 40, so the least possible number of 4s is

$$\boxed{21}.$$

Question 6

(a) Jack wins a round with probability

$$\frac{1}{3}.$$

Hence

$$P(\text{exactly 2 wins}) = \binom{7}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^5 = \boxed{\frac{224}{729}}.$$

(b)(i) Each round has two possible winners, Jack or John. Therefore seven rounds have

$$2^7 = \boxed{128}$$

possible winner-sequences.
(b)(ii) By the binomial theorem,

$$(1+y)^7 = \sum_{k=0}^{7}\binom{7}{k}y^k.$$

Set $y = 1$:

$$\boxed{128 = \sum_{k=0}^{7}\binom{7}{k}}.$$

(b)(iii) The total number of possible winner-sequences equals the sum of the numbers of ways Jack can win exactly k rounds, for $k = 0, 1, \ldots, 7$.
(c)(i) Expected earnings for Jack:

$$7\left[4\left(\frac{1}{3}\right) + (-2)\left(\frac{2}{3}\right)\right] = \boxed{\$0}.$$

Expected earnings for John:

$$7\left[(-4)\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right)\right] = \boxed{\$0}.$$

(c)(ii) Neither player is expected to win.
(c)(iii) Yes. The game is fair because neither player has a positive expected gain.
(d)(i) John wins a round with probability $\frac{2}{3}$.

$$\left[\binom{7}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2\right]\left[\binom{7}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^5\right]$$

$$= \boxed{\binom{7}{5}^2\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^7}.$$

Thus

$$r = 5, \qquad s = 7, \qquad t = 7.$$

(d)(ii)

$$P(\text{John wins 7 of 14}) = \frac{2^7}{3^{14}}\sum_{k=0}^{7}\binom{7}{k}^2.$$

(d)(iii) But directly,

$$P(\text{John wins 7 of 14}) = \binom{14}{7}\frac{2^7}{3^{14}}.$$

Equating gives

$$\boxed{\binom{14}{7} = \sum_{k=0}^{7}\binom{7}{k}^2}.$$

(e)(i) Jack wins each roll with probability $\frac{1}{3}$, so

$$E[A] = \sum_{r=0}^{12} r \binom{12}{r} \frac{2^{12-r}}{3^{12}}.$$

Therefore

$$\boxed{a = 2, \qquad b = 3}.$$

(e)(ii) From

$$(1+y)^{12} = \sum_{r=0}^{12} \binom{12}{r} y^r,$$

differentiate:

$$12(1+y)^{11} = \sum_{r=0}^{12} r \binom{12}{r} y^{r-1}.$$

Set $y = \frac{1}{2}$ and rearrange:

$$\sum_{r=0}^{12} r \binom{12}{r} \frac{2^{12-r}}{3^{12}} = 4.$$

Hence

$$\boxed{E[A] = 4}.$$

Question 7

(a) Let C denote coupon and E denote empty.
For the fourth coupon on the fourth stick:

$$P(X=4)=P(CCCC)=(0.2)^4=\boxed{0.0016}.$$

For the fourth coupon on the fifth stick, exactly one of the first four sticks is empty and the fifth is a coupon:

$$P(X=5)=4(0.8)(0.2)^4=\boxed{0.00512}.$$

(b) Using $n=4$:

$$\frac{1}{3750}(4-1)(4^2+4a+b)=0.0016.$$

Hence

$$4a+b=-14.$$

Using $n=5$:

$$\frac{1}{3750}(5-1)(5^2+5a+b)(0.8)=0.00512,$$

so

$$5a+b=-19.$$

Therefore

$$\boxed{a=-5,\qquad b=6}.$$

(c) Thus

$$\begin{aligned}P(X=n)&=\frac{1}{3750}(n-1)(n^2-5n+6)(0.8)^{n-4}\\&=\frac{1}{3750}(n-1)(n-2)(n-3)(0.8)^{n-4}.\end{aligned}$$

Hence

$$\frac{P(X=n)}{P(X=n-1)}=\boxed{\frac{4(n-1)}{5(n-4)}}.$$

(d)(i) Set the ratio equal to 1:

$$\frac{4(n-1)}{5(n-4)}=1.$$

This gives

$$n=16.$$

The ratio is greater than 1 for $n<16$ and less than 1 for $n>16$, so the distribution has two adjacent modes.
(d)(ii)

$$\boxed{m_1=15,\qquad m_2=16}.$$

(e) Let Y be the number of coupons among N sticks:

$$Y\sim B(N,0.2).$$

Oliver gets at least one free gift when

$$Y\geq 4.$$

We require

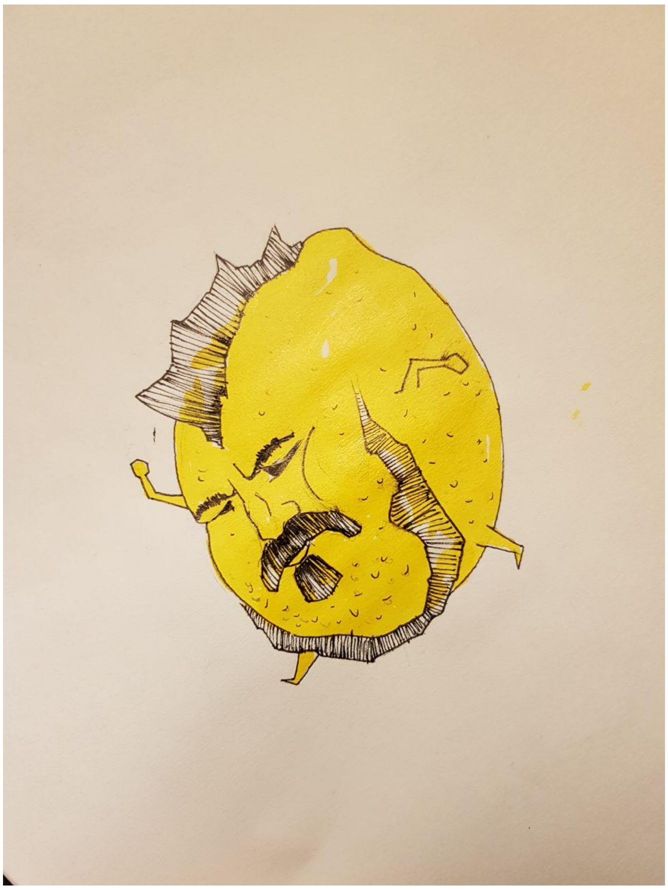
$$P(Y\geq 4)>0.5.$$

Equivalently,

$$P(Y\leq 3)<0.5.$$

Using binomial cumulative probabilities, the least value is

$$\boxed{N=19}.$$



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