DP Analysis & Approaches HL

Maths with Lemon

Inductive Proof Practice

Mathematician: _____

Use mathematical induction to prove each statement for all positive integers. **Summation Proofs**

Question 1. Prove that $6 + 9 + 12 + \cdots + (3n + 3) = \dfrac{n(3n + 9)}{2}$ for all $n \geq 1$.

Question 2. Prove that $45 + 207 + 477 + \cdots + (54n^2 - 9) = 9n^2(2n + 3)$ for all $n \geq 1$.

Question 3. Prove that $20 + 92 + 212 + \cdots + (24n^2 - 4) = 4n^2(2n + 3)$ for all $n \geq 1$.

Question 4. Prove that $16 + 144 + 400 + \cdots + (8n - 4)^2 = \dfrac{16n(4n^2 - 1)}{3}$ for all $n \geq 1$.

Divisibility Proofs

Question 5. Prove that $15n^2 + 9n$ is divisible by 2 for all $n \geq 1$.

Question 6. Prove that $3^n + 3$ is divisible by 2 for all $n \geq 1$.

Question 7. Prove that $n^3 + 2n$ is divisible by 3 for all $n \geq 1$.

Question 8. Prove that $19n^2 - 5n$ is divisible by -2 for all $n \geq 1$.

Inequality Proofs

Question 9. Prove that $8^n \geq 5n$ for all $n \geq 1$.

Question 10. Prove that $4^n \geq 2n$ for all $n \geq 1$.

Question 11. Prove that $8^n \geq 3n$ for all $n \geq 1$.

Question 12. Prove that $6^n \geq 6n$ for all $n \geq 1$.

Detailed Analytical Solutions

Question 1

Statement: $\sum_{r=1}^{n}(3r+3) = \frac{n(3n+9)}{2}$

Anchor Step ($n = 1$)
$\text{LHS} = 6$, $\text{RHS} = \frac{1(3+9)}{2} = 6$.

Inductive Hypothesis ($n = k$)
Assume $\sum_{r=1}^{k}(3r+3) = \frac{k(3k+9)}{2}$.

Inductive Step ($n = k+1$)

$$\begin{aligned}\sum_{r=1}^{k+1}(3r+3) &= \frac{k(3k+9)}{2} + 3(k+1) + 3 \\ &= \frac{3k^2+15k+12}{2} \\ &= \frac{3(k+1)(k+4)}{2} \\ &= \frac{(k+1)(3(k+1)+9)}{2}.\end{aligned}$$

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k+1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 2

Statement: $\sum_{r=1}^{n}(54r^2-9) = 9n^2(2n+3)$

Anchor Step ($n = 1$)
$54(1)^2 - 9 = 45 = 9(1)^2(2+3)$.

Inductive Hypothesis ($n = k$)
Assume $\sum_{r=1}^{k}(54r^2-9) = 9k^2(2k+3)$.

Inductive Step ($n = k+1$)

$$\begin{aligned}\sum_{r=1}^{k+1}(54r^2-9) &= 9k^2(2k+3) + 54(k+1)^2 - 9 \\ &= 18k^3 + 27k^2 + 54k^2 + 108k + 45 \\ &= 18k^3 + 81k^2 + 108k + 45 \\ &= 9(k+1)^2(2k+5) \\ &= 9(k+1)^2(2(k+1)+3).\end{aligned}$$

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k+1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 3

Statement: $\sum_{r=1}^{n}(24r^2-4) = 4n^2(2n+3)$

Anchor Step ($n = 1$)

$24 - 4 = 20 = 4(1)^2(2 + 3)$.

Inductive Hypothesis ($n = k$)
Assume $\sum_{r=1}^{k}(24r^2 - 4) = 4k^2(2k + 3)$.

Inductive Step ($n = k + 1$)

$$\begin{aligned}\sum_{r=1}^{k+1}(24r^2 - 4) &= 4k^2(2k + 3) + 24(k + 1)^2 - 4\\ &= 8k^3 + 12k^2 + 24k^2 + 48k + 20\\ &= 8k^3 + 36k^2 + 48k + 20\\ &= 4(k + 1)^2(2k + 5)\\ &= 4(k + 1)^2(2(k + 1) + 3).\end{aligned}$$

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 4

Statement: $\sum_{r=1}^{n}(8r - 4)^2 = \frac{16n(4n^2-1)}{3}$

Anchor Step ($n = 1$)
$\text{LHS} = 4^2 = 16$, $\text{RHS} = \frac{16(1)(4-1)}{3} = 16$.

Inductive Hypothesis ($n = k$)
Assume $\sum_{r=1}^{k}(8r - 4)^2 = \frac{16k(4k^2-1)}{3}$.

Inductive Step ($n = k + 1$)

$$\begin{aligned}\sum_{r=1}^{k+1}(8r - 4)^2 &= \frac{16k(4k^2 - 1)}{3} + (8(k + 1) - 4)^2\\ &= \frac{16k(4k^2 - 1)}{3} + 16(2k + 1)^2\\ &= \frac{16}{3}\left[4k^3 - k + 3(4k^2 + 4k + 1)\right]\\ &= \frac{16}{3}(4k^3 + 12k^2 + 11k + 3)\\ &= \frac{16}{3}(k + 1)(4k^2 + 8k + 3)\\ &= \frac{16(k + 1)(4(k + 1)^2 - 1)}{3}.\end{aligned}$$

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 5

Statement: $2 \mid (15n^2 + 9n)$

Anchor Step ($n = 1$)
$15(1)^2 + 9(1) = 24 = 2(12)$.

Inductive Hypothesis ($n = k$)

Assume $15k^2 + 9k = 2r$ for some integer r.

Inductive Step ($n = k + 1$)

$$\begin{aligned} 15(k+1)^2 + 9(k+1) &= 15k^2 + 30k + 15 + 9k + 9 \\ &= (15k^2 + 9k) + 30k + 24 \\ &= 2r + 30k + 24 = 2(r + 15k + 12). \end{aligned}$$

Since $r + 15k + 12$ is an integer, the expression is divisible by 2.

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 6

Statement: $2 \mid (3^n + 3)$

Anchor Step ($n = 1$)
$3^1 + 3 = 6 = 2(3)$.

Inductive Hypothesis ($n = k$)
Assume $3^k + 3 = 2r$ for some integer r.

Inductive Step ($n = k + 1$)

$$\begin{aligned} 3^{k+1} + 3 &= 3 \cdot 3^k + 3 \\ &= (2 + 1)3^k + 3 \\ &= 2 \cdot 3^k + (3^k + 3) \\ &= 2 \cdot 3^k + 2r = 2(3^k + r). \end{aligned}$$

Thus the expression is divisible by 2.

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 7

Statement: $3 \mid (n^3 + 2n)$

Anchor Step ($n = 1$)
$1^3 + 2(1) = 3 = 3(1)$.

Inductive Hypothesis ($n = k$)
Assume $k^3 + 2k = 3r$ for some integer r.

Inductive Step ($n = k + 1$)

$$\begin{aligned} (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= (k^3 + 2k) + 3k^2 + 3k + 3 \\ &= 3r + 3k^2 + 3k + 3 \\ &= 3(r + k^2 + k + 1). \end{aligned}$$

Thus the expression is divisible by 3.

Conclusion

The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 8

Statement: $-2 \mid (19n^2 - 5n)$

Anchor Step ($n = 1$)
$19(1)^2 - 5(1) = 14 = (-2)(-7)$.

Inductive Hypothesis ($n = k$)
Assume $19k^2 - 5k = -2r$ for some integer r.

Inductive Step ($n = k + 1$)

$$\begin{aligned} 19(k+1)^2 - 5(k+1) &= 19k^2 + 38k + 19 - 5k - 5 \\ &= (19k^2 - 5k) + 38k + 14 \\ &= -2r + 38k + 14 \\ &= -2(r - 19k - 7). \end{aligned}$$

The quantity in parentheses is an integer, so divisibility by -2 follows.

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 9

Statement: $8^n \geq 5n$

Anchor Step ($n = 1$)
$8^1 = 8 \geq 5 = 5(1)$.

Inductive Hypothesis ($n = k$)
Assume $8^k \geq 5k$ for an integer $k \geq 1$.

Inductive Step ($n = k + 1$)
Multiplying the hypothesis by the positive number 8 gives

$$\begin{aligned} 8^{k+1} &= 8 \cdot 8^k \geq 40k \\ &= 5k + 35k \\ &\geq 5k + 5 \qquad (k \geq 1) \\ &= 5(k+1). \end{aligned}$$

Thus the inequality holds for $n = k + 1$.

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 10

Statement: $4^n \geq 2n$

Anchor Step ($n = 1$)
$4^1 = 4 \geq 2 = 2(1)$.

Inductive Hypothesis ($n = k$)
Assume $4^k \geq 2k$ for an integer $k \geq 1$.

Inductive Step ($n = k + 1$)

Multiplying the hypothesis by the positive number 4 gives

$$\begin{aligned} 4^{k+1} &= 4 \cdot 4^k \geq 8k \\ &= 2k + 6k \\ &\geq 2k + 2 \qquad (k \geq 1) \\ &= 2(k+1). \end{aligned}$$

Thus the inequality holds for $n = k + 1$.

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 11

Statement: $8^n \geq 3n$

Anchor Step ($n = 1$)
$8^1 = 8 \geq 3 = 3(1)$.

Inductive Hypothesis ($n = k$)
Assume $8^k \geq 3k$ for an integer $k \geq 1$.

Inductive Step ($n = k + 1$)
Multiplying the hypothesis by the positive number 8 gives

$$\begin{aligned} 8^{k+1} &= 8 \cdot 8^k \geq 24k \\ &= 3k + 21k \\ &\geq 3k + 3 \qquad (k \geq 1) \\ &= 3(k+1). \end{aligned}$$

Thus the inequality holds for $n = k + 1$.

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.

Question 12

Statement: $6^n \geq 6n$

Anchor Step ($n = 1$)
$6^1 = 6 \geq 6 = 6(1)$.

Inductive Hypothesis ($n = k$)
Assume $6^k \geq 6k$ for an integer $k \geq 1$.

Inductive Step ($n = k + 1$)
Multiplying the hypothesis by the positive number 6 gives

$$\begin{aligned} 6^{k+1} &= 6 \cdot 6^k \geq 36k \\ &= 6k + 30k \\ &\geq 6k + 6 \qquad (k \geq 1) \\ &= 6(k+1). \end{aligned}$$

Thus the inequality holds for $n = k + 1$.

Conclusion
The statement holds for $n = 1$, and its truth for $n = k$ implies its truth for $n = k + 1$. Therefore, by mathematical induction, it holds for all positive integers n.



Maths with Lemon

Mathematics made clear.