DP Analysis & Approaches HL

Maths with Lemon

CN.1 Complex Terminology and Operations

Mathematician: _____

Syllabus: AHL 1.12 and AHL 1.14

AHL 1.12	AHL 1.14
$i^2 = -1$; Cartesian form $z = a + bi$; real and imaginary parts; conjugate, modulus and argument.	Complex conjugate roots of polynomials with real coefficients occur in pairs.

Warm Up

1. Solve $3x^2 - 6x + 2 = 0$. Give your answer in simplest form.

2. Given that 3 and -5 are zeroes of a quadratic polynomial $p(x)$, write a monic $p(x)$ in factorized form.

The Number i and Powers of i

$i = \sqrt{-1}$, so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. The powers repeat in a cycle of four.

$i =$ _____	$i^5 =$ _____	$i^9 =$ _____	$i^{1000} =$ _____
$i^2 =$ _____	$i^6 =$ _____	$i^{10} =$ _____	$i^{67} =$ _____
$i^3 =$ _____	$i^7 =$ _____	$i^{11} =$ _____	$i^{29} =$ _____
$i^4 =$ _____	$i^8 =$ _____	$i^{12} =$ _____	$i^{13} =$ _____

Factored Form with Complex Roots

Factor over the complex numbers, using $a^2+b^2 = (a+bi)(a-bi)$.

Worked Example

$f(x) = x^2 - 9$.

$x^2 - 9 = x^2 - 3^2 = \boxed{(x-3)(x+3)}$.

Worked Example

$f(x) = x^2 + 9$.

$x^2 + 9 = x^2 - (3i)^2 = \boxed{(x-3i)(x+3i)}$.

Your Turn

$f(x) = x^2 + 4$.

Show your working.

.....

.....

.....

Worked Example	Your Turn
$f(x) = x^2 + 3x + 2$. $\boxed{(x+1)(x+2)}$.	$f(x) = x^2 - 2x - 3$. *Show your working.* ....

Worked Example	Your Turn
$f(x) = x^2 + 3x + 3$. The roots are $\dfrac{-3 \pm \sqrt{9-12}}{2} = \dfrac{-3 \pm i\sqrt{3}}{2}$. $\boxed{\left(x + \dfrac{3}{2} - \dfrac{i\sqrt{3}}{2}\right)\left(x + \dfrac{3}{2} + \dfrac{i\sqrt{3}}{2}\right)}$.	$f(x) = x^2 - 2x + 2$. *Show your working.* ....

Complex Numbers and Conjugates

For $z = a + bi$, $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$, and $\overline{z} = a - bi$. The modulus is $|z| = \sqrt{a^2 + b^2}$; its argument is the angle from the positive real axis. For real-coefficient polynomials, non-real roots occur in conjugate pairs. For rational-coefficient polynomials, irrational quadratic-surds also occur in conjugate pairs.

Determining Functions from Roots

Construct a *monic* quadratic $p(x) = (x-r_1)(x-r_2)$ and expand.

Worked Example
Roots $5, -7$: $p(x) = (x-5)(x+7) = \boxed{x^2+2x-35}$.

Worked Example	Your Turn
Root $\sqrt{6}$ (rational coefficients), so $-\sqrt{6}$ is also a root. $p(x) = (x-\sqrt{6})(x+\sqrt{6}) = \boxed{x^2-6}$.	Root $-\sqrt{7}$ (rational coefficients). *Show your working.* ....

Worked Example	Your Turn
Root $4-\sqrt{6}$; partner $4+\sqrt{6}$. $p(x) = (x-4)^2 - 6 = \boxed{x^2-8x+10}$.	Root $2-\sqrt{3}$ (rational coefficients). *Show your working.* ....

Worked Example	Your Turn
Root $5-3i$; partner $5+3i$. $p(x) = (x-5)^2 + 9 = \boxed{x^2-10x+34}$.	Root $3+2i$ (real coefficients). *Show your working.* ....

Addition and Subtraction of Complex Numbers

Worked Example	Your Turn
$z = 2+i$, $w = 5-3i$. $z+w = \boxed{7-2i}$ $z-w = \boxed{-3+4i}$ $w-z = \boxed{3-4i}$.	$z = 7+11i$, $w = 4-i$. Find $z+w$, $z-w$, $w-z$. *Show your working.* ....

Multiplication of Complex Numbers

Expand and use $i^2 = -1$.

Worked Example	Your Turn
$z = 2 + i$, $w = 5 - 3i$. $zw = 10 - 6i + 5i - 3i^2$ $= \boxed{13 - i}$.	$z = 7 + 11i$, $w = 4 - i$. Find zw. *Show your working.* ....

Worked Example	Your Turn
$z = 4 - 6i$, $w = -3 + 5i$. $zw = -12 + 20i + 18i - 30i^2$ $= \boxed{18 + 38i}$.	$z = 2\sqrt{2} + 2i\sqrt{2}$, $w = i\sqrt{2}$. Find zw. *Show your working.* ....

Division of Complex Numbers

Multiply numerator and denominator by the conjugate of the denominator.

Worked Example
$\frac{5+4i}{4+3i}$. $\frac{(5+4i)(4-3i)}{(4+3i)(4-3i)} = \frac{32+i}{25} = \boxed{\frac{32}{25} + \frac{1}{25}i}$

Worked Example	Your Turn
$\frac{2+5i}{-6i}$. $\frac{(2+5i)i}{(-6i)i} = \frac{-5+2i}{6} = \boxed{-\frac{5}{6} + \frac{1}{3}i}$.	$-\frac{i}{-3-i}$. *Show your working.* ....

Complex Number Equations

Write $z = a+bi$, $\overline{z} = a-bi$; compare real and imaginary parts.

Worked Example

$5z + 3\overline{z} = 8 - 4i$.

$$5(a + bi) + 3(a - bi) = 8a + 2bi.$$

Thus $8a = 8$, $2b = -4$; $\boxed{z = 1 - 2i}$.

Worked Example	Your Turn
$4z - 3\overline{z} = \dfrac{1 - 18i}{2 - i}$. $\dfrac{(1 - 18i)(2 + i)}{5} = 4 - 7i$. Left side $= a + 7bi$; hence $\boxed{z = 4 - i}$.	$2z - 3\overline{z} = \dfrac{-27 + 23i}{1 + i}$. *Show your working.* ....

Worked Example	Your Turn
$z = -3 + 4i$, $zw = -14 + 2i$. $w = \dfrac{-14 + 2i}{-3 + 4i} = \dfrac{50 + 50i}{25} = \boxed{2 + 2i}$.	$z = 22 + 4i$, $\dfrac{z}{w} = 6 - 8i$. Find w. *Show your working.* ....

Worked Example	Your Turn
$x(1 + i)^2 + y(2 - i)^2 = 3 + 10i$. $(1 + i)^2 = 2i, \quad (2 - i)^2 = 3 - 4i$. $3y + (2x - 4y)i = 3 + 10i$. $y = 1$, $2x - 4 = 10$; $\boxed{x = 7,\ y = 1}$.	$(x + iy)(3 + 4i) = 3 - 4i$. Find real x, y. *Show your working.* ....

Complex Number Equations – Continued

Worked Example	Your Turn
$z^2 = 21 - 20i$. Let $z = a + bi$. $$a^2 - b^2 = 21, \quad 2ab = -20.$$ Since $\lvert z^2\rvert = 29$, $a^2 + b^2 = 29$. Thus $a^2 = 25$, $b^2 = 4$, $ab = -10$. $$\boxed{z = 5 - 2i \quad \text{or} \quad z = -5 + 2i}.$$	$z - 8 = i(7 - 2\overline{z})$. *Show your working.* ....

Worked Example	Your Turn
$z - 12 = i(9 - 2\overline{z})$. Set $z = a + bi$. $$a - 12 + bi = i(9 - 2a + 2bi).$$ Right side $= -2b + (9 - 2a)i$. So $a - 12 = -2b$, $b = 9 - 2a$. Solving gives $\boxed{z = 2 + 5i}$.	$2z - i\overline{z} = 3(3 - 5i)$. *Show your working.* ....

Roots of Polynomials

If a polynomial has real coefficients and $a+bi$ is a root, $a-bi$ is also a root.

Worked Example
$p(x) = x^3 - 5x^2 + 11x - 15$ and $x = 3$ is Synthetic division by $x-3$ gives quotient x^2 $$\boxed{p(x) = (x - 3)(x^2 - 2x + 5)}.$$ The remaining roots are $1 \pm 2i$.

Detailed Markscheme

CN.1 Complex Terminology and Operations

Warm Up

1. $x = \frac{6 \pm \sqrt{36-24}}{6} = \boxed{1 \pm \frac{\sqrt{3}}{3}}$.

2. $p(x) = \boxed{(x-3)(x+5)}$ (monic form).

Powers of i

	1	2	3	4
row 1	i	i	i	1
row 2	-1	-1	-1	$-i$
row 3	$-i$	$-i$	$-i$	i
row 4	1	1	1	i

Columns correspond to exponents $(1, 5, 9, 1000)$, $(2, 6, 10, 67)$, $(3, 7, 11, 29)$ and $(4, 8, 12, 13)$ respectively.

Factored Form – Your Turn

$x^2 - 4$**:** $\boxed{(x-2)(x+2)}$.

$x^2 + 4$**:** $\boxed{(x-2i)(x+2i)}$.

$x^2 - 2x - 3$**:** $\boxed{(x-3)(x+1)}$.

$x^2 - 2x + 2$**:** Roots $1 \pm i$, so $\boxed{(x-1-i)(x-1+i)}$.

Determining Functions from Roots – Your Turn

Roots $2, -8$**:** $\boxed{p(x) = (x-2)(x+8) = x^2 + 6x - 16}$.

Root $-\sqrt{7}$**:** Conjugate root $\sqrt{7}$; $\boxed{p(x) = x^2 - 7}$.

Root $2 - \sqrt{3}$**:** Conjugate $2 + \sqrt{3}$; $p(x) = (x-2)^2 - 3 = \boxed{x^2 - 4x + 1}$.

Root $3 + 2i$**:** Conjugate $3 - 2i$; $p(x) = (x-3)^2 + 4 = \boxed{x^2 - 6x + 13}$.

Addition and Subtraction – Your Turn

For $z = 7 + 11i$, $w = 4 - i$:

$$\boxed{z + w = 11 + 10i, \quad z - w = 3 + 12i, \quad w - z = -3 - 12i}.$$

Multiplication – Your Turn

$$(7 + 11i)(4 - i) = 28 - 7i + 44i - 11i^2 = \boxed{39 + 37i},$$
$$(2\sqrt{2} + 2i\sqrt{2})(i\sqrt{2}) = 4i + 4i^2 = \boxed{-4 + 4i}.$$

Division – Your Turn

$$\frac{5 + 4i}{2 - 2i} = \frac{(5 + 4i)(2 + 2i)}{8} = \frac{2 + 18i}{8} = \boxed{\frac{1}{4} + \frac{9}{4}i},$$
$$-\frac{i}{-3 - i} = \frac{-i(-3 + i)}{10} = \boxed{\frac{1}{10} + \frac{3}{10}i}.$$

Complex Number Equations – Your Turn

1. $z/(z+2) = 2-i$ implies $z = (2-i)(z+2)$, hence $(-1+i)z = 4-2i$ and $\boxed{z = -3-i}$.

2. $(-27+23i)/(1+i) = (-4+50i)/2 = -2+25i$. For $z = a+bi$, $2z-3\overline{z} = -a+5bi$; thus $\boxed{z = 2+5i}$.

3. $w = z/(6-8i) = (22+4i)(6+8i)/100 = (100+200i)/100 = \boxed{1+2i}$.

4. $3x-4y = 3$, $4x+3y = -4$. Multiplying gives $x = (9-16)/25 = \boxed{-7/25}$ and $y = (-12-12)/25 = \boxed{-24/25}$.

5. For $z = a+bi$, $z-8 = i(7-2\overline{z}) = -2b+(7-2a)i$. So $a-8 = -2b$, $b = 7-2a$. Solve $a-8 = -14+4a$, giving $a = 2$, $b = 3$. Thus $\boxed{z = 2+3i}$.

6. $2z-i\overline{z} = (2a-b)+(2b-a)i = 9-15i$. Solve $2a-b = 9$, $2b-a = -15$, giving $\boxed{z = 1-7i}$.

Roots of Polynomials – Your Turn

Since $3-2i$ is a root of the real-coefficient polynomial, so is $3+2i$.

$$(x-(3-2i))(x-(3+2i)) = (x-3)^2+4 = x^2-6x+13.$$

Divide x^3-4x^2+x+26 by $x^2-6x+13$ to obtain $x+2$:

$$p(x) = (x+2)(x^2-6x+13).$$

Therefore the roots are $\boxed{-2,\ 3-2i,\ 3+2i}$.



Maths with Lemon

Mathematics made clear.