DP Analysis & Approaches HL

Maths with Lemon

CN.3 Powers and Roots of Complex Numbers

Mathematician: _____

AHL 1.14 – De Moivre's Theorem and Complex Roots

Syllabus	Guidance and links
De Moivre's theorem, including rational exponents; powers and roots of complex numbers.	Proof by induction for positive integers; links to polynomial roots (AHL 2.12) and compound-angle identities (AHL 3.10).

Warm Up

Let $z = 2\,\text{cis}(2\theta)$ where $0 < \theta < 45°$. Find the modulus and argument of $z + 2$ in terms of θ. [6]

Show your working.

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Show your working.

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Investigate

Let $z = r\,\text{cis}\,\theta$. Use multiplication in polar form to obtain z^2, z^3, z^4, z^5 and conjecture z^n.

$z^2 =$ _____ $z^3 =$ _____

$z^4 =$ _____ $z^5 =$ _____

Conjecture: _____

The Big Picture

Find all complex solutions of $x^4 = 16$.

Show your working.

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Powers in Rectangular Form

Worked Example	You Try
1. $(1+i)^2$ $(1+i)^2 = 1+2i+i^2 = \boxed{2i}$.	**1.** $(1+i)^3$ *Show your working.* ....

Worked Example	You Try
2. $(4-6i)^3$ $(4-6i)^2 = -20-48i$, hence $(-20-48i)(4-6i) = \boxed{-368-72i}$.	**2.** $(3-3i)^3$ *Show your working.* ....

Worked Example	You Try
3. $\left(\frac{5}{2} - \frac{5\sqrt{3}}{2}i\right)^4$ The modulus is 5 and argument $-\pi/3$. Thus $5^4 \operatorname{cis}(-4\pi/3) = 625 \operatorname{cis}(2\pi/3)$ $= \boxed{-\frac{625}{2} + \frac{625\sqrt{3}}{2}i}$.	**3.** $(-\sqrt{3}-i)^4$ *Show your working.* ....

De Moivre's Theorem

For $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{Z}$ (with $z \neq 0$ when $n < 0$),

$$\boxed{z^n = r^n(\cos n\theta + i\sin n\theta) = r^n \operatorname{cis}(n\theta)}.$$

Powers in Polar Form

Worked Example	You Try
4. $[4\,\mathrm{cis}(5\pi/4)]^5$ $= 4^5\,\mathrm{cis}(25\pi/4) = \boxed{1024\,\mathrm{cis}(\pi/4)}$.	**4.** $[4\,\mathrm{cis}\,270°]^4$ *Show your working.* ....

Worked Example	You Try
5. $[3\,\mathrm{cis}\,135°]^5$ $= 3^5\,\mathrm{cis}\,675° = \boxed{243\,\mathrm{cis}\,315°}$.	**5.** $[4\,\mathrm{cis}\,90°]^4$ *Show your working.* ....

Worked Example	You Try
6. $[6\,\mathrm{cis}\,315°]^3$ $= 6^3\,\mathrm{cis}\,945° = \boxed{216\,\mathrm{cis}\,225°}$.	**6.** $[2\,\mathrm{cis}(7\pi/6)]^3$ *Show your working.* ....

Negative Integer Exponents – Cartesian Answers

Worked Example	You Try
7. $[3\,\mathrm{cis}\,330°]^{-5}$ $= \frac{1}{243}\,\mathrm{cis}(-1650°) = \frac{1}{243}\,\mathrm{cis}\,150°$ $= \boxed{-\frac{\sqrt{3}}{486} + \frac{i}{486}}$.	**7.** $[3\,\mathrm{cis}\,315°]^{-3}$ *Show your working.* ....

Worked Example	You Try
8. $[5\,\mathrm{cis}(\pi/3)]^{-5}$ $= \frac{1}{3125}\,\mathrm{cis}(-5\pi/3) = \frac{1}{3125}\,\mathrm{cis}(\pi/3)$ $= \boxed{\frac{1}{6250} + \frac{\sqrt{3}}{6250}i}.$	**8.** $[6\,\mathrm{cis}(5\pi/4)]^{-4}$ *Show your working.* ....

Combining Skills

Worked Example	You Try
Problem 1. Write $\dfrac{1+i}{(1-i)^5}$ in polar form. $1+i=\sqrt{2}\,\mathrm{cis}(\pi/4)$, $1-i=\sqrt{2}\,\mathrm{cis}(-\pi/4)$. $$\frac{\sqrt{2}\,\mathrm{cis}(\pi/4)}{(\sqrt{2})^5\,\mathrm{cis}(-5\pi/4)}=\frac{1}{4}\,\mathrm{cis}(3\pi/2)$$ $\boxed{\frac{1}{4}\,\mathrm{cis}(3\pi/2)}$.	**Problem 2.** If $w=\sqrt{3}-i$, find $\dfrac{1}{w^2}$ in polar form. *Show your working.* ....

Roots of Complex Numbers

For $z^n = r\,\mathrm{cis}\,\theta$ with $r>0$, all n roots are

$$\boxed{z_k=\sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta+2\pi k}{n}\right),\quad k=0,1,\ldots,n-1}.$$

The roots lie equally spaced on a circle of radius $\sqrt[n]{r}$.

Worked Example	
9. Find all cube roots of 1 in Cartesian form. $1=\mathrm{cis}\,0$, so $z_k=\mathrm{cis}(2\pi k/3)$, $k=0,1,2$. $\boxed{1,\quad -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i}$.	**9.** Solve *Show you* ....

Worked Example	You Try
10. Find the cube roots of $8i$. $8i=8\,\mathrm{cis}(\pi/2)$, so $$z_k=2\,\mathrm{cis}\left(\frac{\pi}{6}+\frac{2\pi k}{3}\right).$$ $\boxed{\sqrt{3}+i,\quad -\sqrt{3}+i,\quad -2i}$.	**10.** Find the cube roots of $-6i$. *Show your working.* ....

Finding All nth Roots

Worked Example	You Try
11. Find the fifth roots of $4\,\mathrm{cis}\,315°$. $$z_k = \sqrt[5]{4}\,\mathrm{cis}\left(\frac{315° + 360°k}{5}\right)$$ $$= \boxed{\sqrt[5]{4}\,\mathrm{cis}(63° + 72°k)},$$ $k = 0, 1, 2, 3, 4$; angles $63°, 135°, 207°, 279°, 351°$.	**11.** Find both square roots of $4\,\mathrm{cis}\,60°$. *Show your working.* ....

Worked Example	You Try
12. Find the fourth roots of $\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$. $z = 5\,\mathrm{cis}\,45°$; hence $$w_k = \sqrt[4]{5}\,\mathrm{cis}\left(\frac{45° + 360°k}{4}\right)$$ $$= \boxed{\sqrt[4]{5}\,\mathrm{cis}(11.25° + 90°k)},$$ $k = 0, 1, 2, 3$.	**12.** Find all fourth roots of $2\sqrt{2} + 2\sqrt{2}i$. *Show your working.* ....

Extending Concepts: How Many Distinct Positive Integer Powers?

Important: If $|z| \neq 1$ and $z \neq 0$, then $|z^n| = |z|^n$ changes with n: there are **infinitely many** distinct powers.

If $|z| = 1$, count the smallest positive m for which $z^m = 1$.

Worked Example	
13. $z = -\frac{\sqrt{2}}{2}(1 + i)$ $z = \mathrm{cis}(5\pi/4)$. Since $5/8$ is reduced, the smallest m with $5m/8 \in \mathbb{Z}$ is $m = 8$. $\boxed{8 \text{ distinct powers}}$.	**13.** *Sho* ....

Worked Example	You Try
14. $z = 4\,\text{cis}\,315°$ $\lvert z\rvert = 4 \neq 1$. The moduli $4, 4^2, 4^3, \ldots$ are distinct. Infinitely many distinct powers.	**14.** $z = 4\,\text{cis}\,60°$ *Show your working.* ....

Worked Example	You Try
15. $z = \sqrt{3} - i$ $\lvert z\rvert = \sqrt{3+1} = 2 \neq 1$, so $\lvert z^n\rvert = 2^n$. Infinitely many distinct powers.	**15.** $z = -2\sqrt{3} + 2i$ *Show your working.* ....

Detailed Analytical Markscheme

Warm Up and Investigation

$z+2 = 2(\cos 2\theta + 1 + i\sin 2\theta) = 4\cos\theta(\cos\theta + i\sin\theta)$. Since $0 < \theta < 45°$, $\cos\theta > 0$. Therefore

$$\boxed{|z+2| = 4\cos\theta, \qquad \arg(z+2) = \theta}.$$

Investigation: $z^2 = r^2\operatorname{cis} 2\theta$, $z^3 = r^3\operatorname{cis} 3\theta$, $z^4 = r^4\operatorname{cis} 4\theta$, $z^5 = r^5\operatorname{cis} 5\theta$; hence $z^n = r^n\operatorname{cis} n\theta$. For $x^4 = 16$: $x = 2\operatorname{cis}(k\pi/2)$, $k = 0,1,2,3$; $\boxed{x = 2, 2i, -2, -2i}$.

Rectangular Powers – Student Questions 1–3

1. $(1+i)^2 = 2i$; $(1+i)^3 = 2i(1+i) = \boxed{-2+2i}$.

2. $(3-3i)^2 = -18i$; $(3-3i)^3 = (-18i)(3-3i) = \boxed{-54-54i}$.

3. $-\sqrt{3} - i = 2\operatorname{cis} 210°$; fourth power $= 16\operatorname{cis} 840° = 16\operatorname{cis} 120° = \boxed{-8+8\sqrt{3}i}$.

Polar Powers – Student Questions 4–6

4. $4^4\operatorname{cis}(4\cdot 270°) = 256\operatorname{cis} 1080° = \boxed{256\operatorname{cis} 0°}$.

5. $4^4\operatorname{cis} 360° = \boxed{256\operatorname{cis} 0°}$.

6. $2^3\operatorname{cis}(3\cdot 7\pi/6) = 8\operatorname{cis}(7\pi/2) = \boxed{8\operatorname{cis}(3\pi/2)}$.

Negative Powers – Student Questions 7–8

7. $3^{-3}\operatorname{cis}(-945°) = \frac{1}{27}\operatorname{cis} 135° = \boxed{-\frac{\sqrt{2}}{54} + \frac{\sqrt{2}}{54}i}$.

8. $6^{-4}\operatorname{cis}(-5\pi) = \frac{1}{1296}\operatorname{cis}\pi = \boxed{-\frac{1}{1296}}$.

Combining Skills – Problem 2

$w = \sqrt{3} - i = 2\operatorname{cis}(-\pi/6)$, hence

$$\frac{1}{w^2} = \frac{1}{4\operatorname{cis}(-\pi/3)} = \boxed{\frac{1}{4}\operatorname{cis}(\pi/3)}.$$

Roots – Student Questions 9–10

9. $-1 = \operatorname{cis}\pi$; fifth roots are

$$\boxed{z_k = \operatorname{cis}\left(\frac{\pi + 2\pi k}{5}\right),\ k = 0,1,2,3,4}.$$

Their arguments are $\pi/5, 3\pi/5, \pi, 7\pi/5, 9\pi/5$.

10. $-6i = 6\operatorname{cis}(3\pi/2)$. The cube roots are

$$\boxed{z_k = \sqrt[3]{6}\operatorname{cis}\left(\frac{\pi}{2} + \frac{2\pi k}{3}\right),\ k = 0,1,2}.$$

Angles: $\pi/2, 7\pi/6, 11\pi/6$.

Further Roots – Student Questions 11–12

11. $4\operatorname{cis} 60°$ has square roots

$$\boxed{2\operatorname{cis} 30°, \qquad 2\operatorname{cis} 210°}.$$

12. $2\sqrt{2} + 2\sqrt{2}i = 4\operatorname{cis} 45°$, so

$$\boxed{z_k = \sqrt{2}\operatorname{cis}(11.25° + 90°k), \quad k = 0,1,2,3}.$$

Extending Concepts – Student Questions 13–15

13. $z = 5\,\mathrm{cis}(\pi/4)$, so $|z^n| = 5^n$; $\boxed{\text{infinitely many}}$.

14. $z = 4\,\mathrm{cis}\,60°$; $|z^n| = 4^n$; $\boxed{\text{infinitely many}}$.

15. $|-2\sqrt{3} + 2i| = 4$, so $|z^n| = 4^n$; $\boxed{\text{infinitely many}}$.

Teaching note: Counting a finite number of distinct arguments modulo 2π is a different question from counting distinct complex powers. The latter is infinite whenever $|z| \neq 1$ (for nonzero z).



Maths with Lemon

Mathematics made clear.