Graphs of Polynomials

Student Worksheet – Markscheme at the End

Maths with Lemon

Mathematician: _____

Warm-up: The discriminant

Using each digit 0–9 at most once across the nine coefficient boxes, construct three quadratics $ax^2 + bx + c$: one with no real roots, one with one repeated real root, and one with two distinct real roots. Explain using $b^2 - 4ac$.

Multiplicity and sketching

1. Explain how a graph behaves at a root of multiplicity (a) 1, (b) 2 or any even multiplicity, (c) 3 or any larger odd multiplicity.

2. For $f(x) = (x+4)^a(x-2)^b(x+1)^c$, sketch the graph in each case, showing roots and end behaviour:

(a) $a = b = c = 1$.

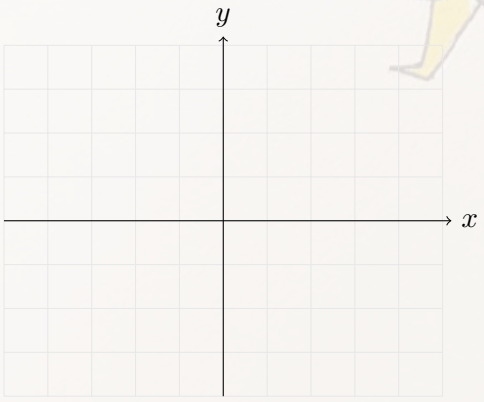
(b) $a = 2, b = c = 1$.

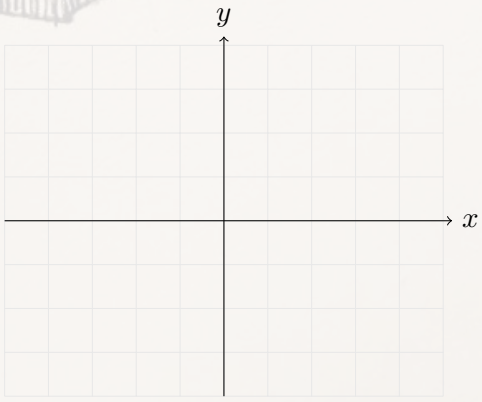
(c) $a = 3, b = c = 1$.

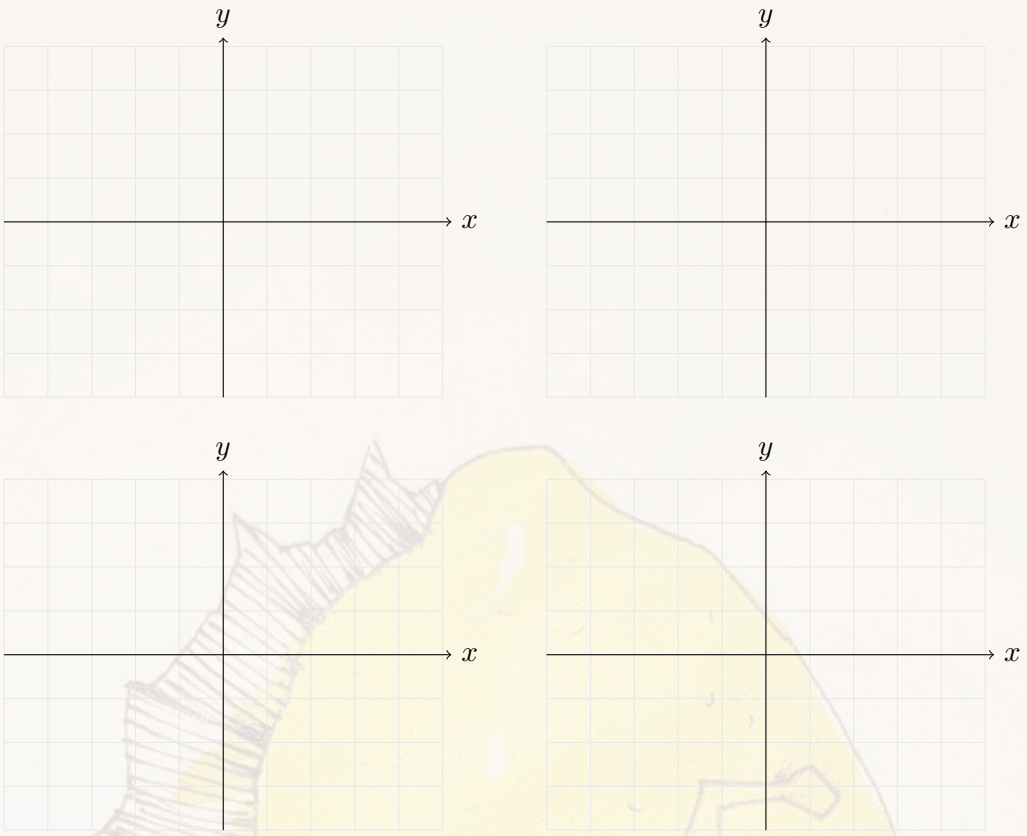
(d) $a = 1, b = 2, c = 1$.

(e) $a = 1, b = 3, c = 1$.

(f) $a = b = 1, c = 2$.







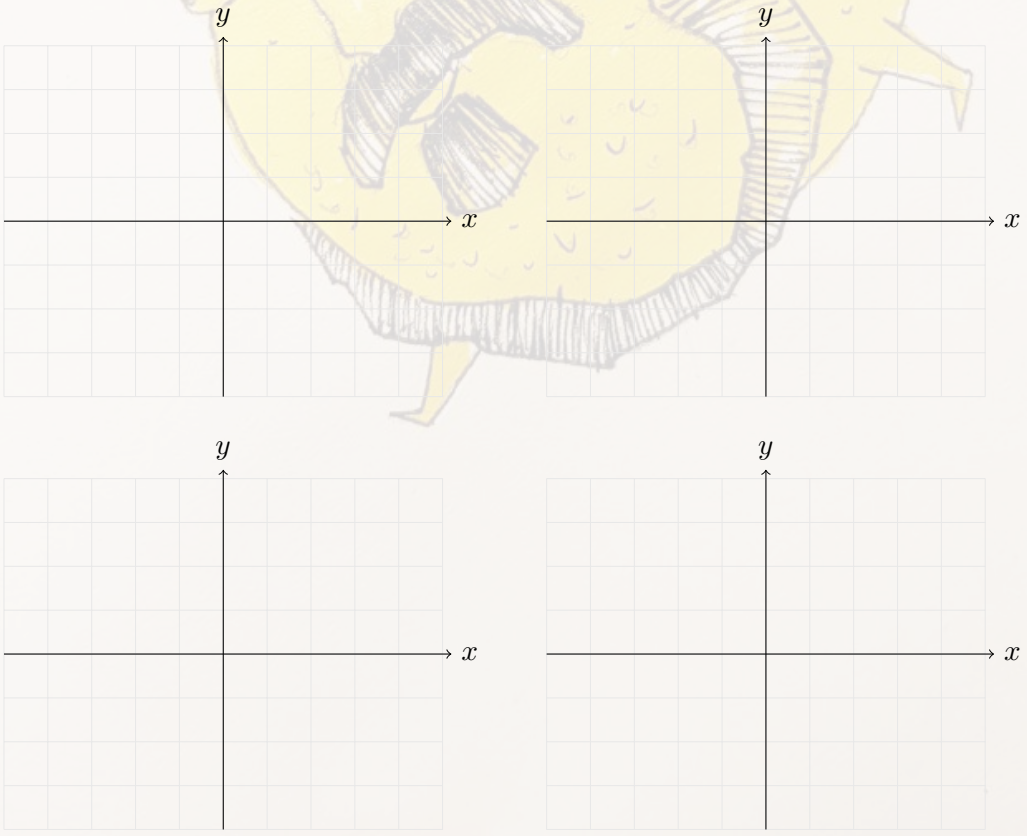
3. Sketch the following. Label roots, multiplicities and end behaviour.

(a) $x^2(x+1)(x-2)$

(b) $x^3(x-1)(x+2)$

(c) $(x-3)^3(x-1)^2(x+2)^2$

(d) $x(x-2)^3(x+2)^2$.



Writing functions from graphs

4. For each description, give a possible minimum-degree polynomial (up to a nonzero constant), its degree and end behaviour.

(a) Cross at -1, touch at 0, cross with flattening at 2; both ends up.

(b) Cross with flattening at -2, cross at 0, touch at 4; both ends down.

(c) Touch at -1, cross at 1, touch at 3; left end down, right end up.

These descriptions reproduce the labelled root behaviour in the three source graphs, rather than relying on a raster image.

Factor theorem and conjugate roots

5. Factor $x^3 + x^2 - 9x - 9$ completely.

6. Factor $x^3 - 2x^2 - 5x + 6$ completely using the factor theorem.

7. Factor $x^3 + x^2 - 17x + 15$ completely.

8. Write a polynomial with rational coefficients, in factored form without radicals or i, having the stated roots. Give the minimum degree.

(a) $5, -7$

(b) $4, \sqrt{6}$

(c) $9, -5i$

(d) $3, -\sqrt{2}, 2i$.

Factorisation from graph information

For each polynomial, use the displayed graph roots from the original lesson, then factor algebraically (synthetic division is recommended). **9.** $x^6 - 4x^5 - 17x^4 + 30x^3 - 70x^2 + 700x + 2000$. Find all roots and classify non-rational roots.

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10. $x^6+4x^5-37x^4-110x^3+470x^2+700x-2000$. The graph shows real roots approximately $-5, -3.162, 2, 3.162, 4$. Factor and classify all roots.

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11. $x^6 - 27x^4 + 8x^3 + 42x^2 + 1440$. The graph shows real roots $-5, -3, 4$ (touching at 4). Factor and classify all roots.

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12. Given that -2 and $1/3$ are roots of $3x^4 + 5x^3 + x^2 + 5x - 2$, factor completely.

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Additional problems

13. Given $x - 2$ is a factor of $x^3 - kx^2 - 24x + 28$, find k, all roots and the full factorisation.

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14. Find the quadratic with roots $-1, \frac{1}{3}$ and value 10 at $x = 2$.

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15. Find the cubic with root -1, double root 3, and value 12 at $x = 2$.

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16. Cinema: $F(x) = 6x^3 + 53x^2 + 72x + 90$ viewers and $G(x) = 2x + 15$ movies. Simplify the average viewers per movie.

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17. A rectangular solid has volume $3x^4 - 3x^3 - 33x^2 + 54x$, length $3x$, width $x - 2$. Find its height.

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18. A rectangle has area $3x^3 + 14x^2 - 23x + 6$, width $x + 6$. Find its length.

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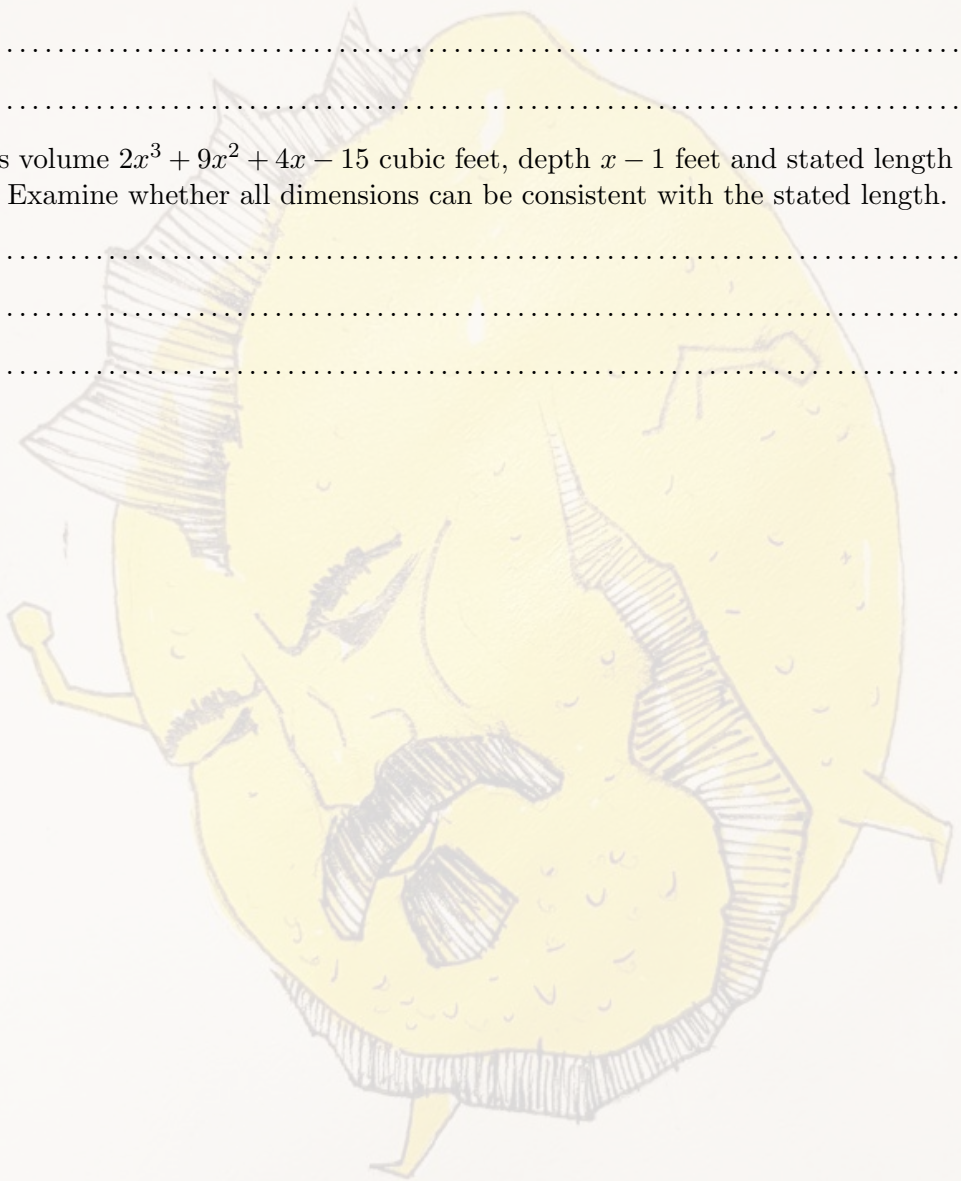
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19. A tank has volume $2x^3 + 9x^2 + 4x - 15$ cubic feet, depth $x - 1$ feet and stated length 13 feet. (a) Factor the volume. (b) Examine whether all dimensions can be consistent with the stated length.

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Detailed Markscheme

Warm-up

One valid assignment using the nine distinct digits $2, 0, 4, 1, 6, 9, 3, 8, 5$ is:

$$2x^2 + 0x + 4: \ D = 0 - 32 = -32 < 0; \qquad x^2 + 6x + 9: \ D = 36 - 36 = 0;$$

$$3x^2 + 8x + 5: \ D = 64 - 60 = 4 > 0.$$

Thus these have no real roots, one repeated real root and two distinct real roots respectively.

1. Multiplicity

Simple root: cross. Even multiplicity: touch and turn. Odd multiplicity ≥ 3: cross with flattening.

2. Six sketches: exact information

Each case has roots $-4, 2, -1$, with multiplicities (a, b, c). The degree is $a + b + c$ and leading coefficient $+1$. Thus (a) degree 3: all cross, left down/right up; (b) degree 4: touch at -4, cross at $-1, 2$, both up; (c) degree 5: flattened crossing at -4, other roots cross, left down/right up; (d) degree 4: touch at 2, others cross, both up; (e) degree 5: flattened crossing at 2, others cross, left down/right up; (f) degree 4: touch at -1, others cross, both up.

3. Four sketches

(a) Degree 4, both ends up: cross $-1, 2$; touch 0. (b) Degree 5, left down/right up: cross $-2, 1$; flattened cross 0. (c) Degree 7, left down/right up: touch $-2, 1$, flattened cross 3. (d) Degree 6, both ends up: touch -2, cross 0, flattened cross 2.

4. Functions from graph descriptions

(a) $\boxed{x^2(x+1)(x-2)^3}$, degree 6, both ends up. (b) $\boxed{-x(x+2)^3(x-4)^2}$, degree 6, both ends down. (c) $\boxed{a(x+1)^2(x-1)(x-3)^2, \ a > 0}$; choose $+(x+1)^2(x-1)(x-3)^2$, degree 5, left down/right up.

5–8. Factorisation

5. $\boxed{(x-3)(x+1)(x+3)}$.

6. $\boxed{(x-3)(x-1)(x+2)}$.

7. $\boxed{(x-3)(x-1)(x+5)}$.

8(a) $(x-5)(x+7)$, degree 2. (b) $(x-4)(x^2-6)$, degree 3. (c) $(x-9)(x^2+25)$, degree 3. (d) $(x-3)(x^2-2)(x^2+4)$, degree 5.

9–12. Factorisation from graph information

9. $P(x) = (x-5)^2(x+2)(x+4)\left(x^2+10\right)$.

10. $P(x) = (x-4)(x-2)(x+5)^2\left(x^2-10\right)$.

11. $P(x) = x^6 - 27x^4 + 8x^3 + 42x^2 + 1440$.

12. $P(x) = (x+2)(3x-1)\left(x^2+1\right)$.

13–19. Additional problems

13. $P(2) = 8 - 4k - 48 + 28 = 0 \Rightarrow k = -3$. $P(x) = (x-2)^2(x+7)$, roots 2 (double), -7.

14. $P(x) = a(x+1)(x-\frac{1}{3})$. $P(2) = 5a = 10 \Rightarrow a = 2$. Thus $P(x) = 2(x+1)(x-\frac{1}{3})$.

15. $P(x) = a(x+1)(x-3)^2$. $P(2) = 3a = 12 \Rightarrow a = 4$. Thus $P(x) = 4(x+1)(x-3)^2$. **16.** $\dfrac{6x^3 + 53x^2 + 72x + 90}{2x + 15} = 3x^2 + 4x + 6$.

17. $\dfrac{3x^4 - 3x^3 - 33x^2 + 54x}{3x(x-2)} = x^2 + x - 9$.

18. $\dfrac{3x^3 + 14x^2 - 23x + 6}{x + 6} = 3x^2 - 4x + 1$.

19. $V(x) = (x-1)(x+3)(2x+5)$. Since the depth is $x - 1$, the other factors are $x + 3$ and $2x + 5$. None is the constant 13; the stated length 13 does not follow from the polynomial factorisation. If length really is 13, the width is $\frac{(x+3)(2x+5)}{13}$. The information does not determine a unique numerical x or volume. This is an inconsistency/underdetermined part of the source question.



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Mathematics made clear.