Graphs of Polynomials

Theory and Worked Examples – AHL 2.12

Maths with Lemon

1. Zeros, factors and multiplicity

A polynomial $P(x) = a_n x^n + \cdots + a_0$ has degree n, where $a_n \neq 0$. A zero r satisfies $P(r) = 0$; therefore $x - r$ is a factor. If $P(x) = (x-r)^m Q(x)$ and $Q(r) \neq 0$, the root r has **multiplicity** m.

- Odd multiplicity: graph **crosses** the x-axis; for $m \geq 3$ it flattens.
- Even multiplicity: graph **touches and turns** at the x-axis.
- Higher multiplicity: flatter behaviour near the root.

$$(x+1)(x-2)^2(x+3)^3$$

has roots -1 (simple crossing), 2 (touching), and -3 (flattened crossing). Its degree is $1+2+3=6$.

2. End behaviour

The leading term $a_n x^n$ determines the ends of the graph.

Degree	$a_n > 0$	$a_n < 0$
Even	both ends up	both ends down
Odd	left down, right up	left up, right down

For example, if $P(x) = x^2(x+1)(x-2)$, the leading term is x^4: $\lim_{x\to-\infty} P(x) = +\infty$ and $\lim_{x\to+\infty} P(x) = +\infty$.

3. Sketching a polynomial from its factors

1. Read the roots from each factor and label the multiplicities.
2. Determine the degree and leading coefficient.
3. Determine the end behaviour.
4. Find the y-intercept $P(0)$.
5. Cross at odd-multiplicity roots and turn at even-multiplicity roots.

Worked example: $P(x) = x^2(x+1)(x-2)$. Roots: -1 (simple), 0 (double), 2 (simple). Degree 4, positive leading coefficient; both ends up. At $x = -1$ and $x = 2$, cross; at $x = 0$, touch.

4. Writing an equation from a graph

If the intercepts are $r_1, \ldots, r_k$, write

$$P(x) = a(x-r_1)^{m_1} \cdots (x-r_k)^{m_k}.$$

Choose the smallest multiplicities consistent with the crossings and touches to obtain a **minimum-degree** polynomial. A known point determines a. **Example:** Cross at -1, touch at 2, and $P(0) = 8$:

$$P(x) = a(x+1)(x-2)^2, \quad 8 = 4a, \quad \boxed{P(x) = 2(x+1)(x-2)^2}.$$

5. Factor theorem and full factorisation

The **Factor Theorem** says $x - r$ is a factor of $P(x)$ iff $P(r) = 0$. After finding a root, use synthetic or polynomial division to reduce the degree.

$$P(x) = x^3 + x^2 - 9x - 9 = x^2(x+1) - 9(x+1) = (x+1)(x^2-9) = \boxed{(x+1)(x-3)(x+3)}.$$

For $x^3 - 2x^2 - 5x + 6$, $P(1) = 0$; division by $x - 1$ gives $x^2 - x - 6 = (x-3)(x+2)$. Thus

$$\boxed{P(x) = (x-1)(x-3)(x+2)}.$$

6. Irrational and complex conjugate roots

For polynomials with **rational coefficients**, irrational quadratic-conjugate roots $a+\sqrt{b}$ and $a-\sqrt{b}$ occur together when a, b are rational and $\sqrt{b}$ is irrational. Their factor is

$$(x-a-\sqrt{b})(x-a+\sqrt{b}) = (x-a)^2 - b.$$

For polynomials with **real coefficients**, nonreal roots $a+bi$ and $a-bi$ occur in conjugate pairs, giving

$$(x-a-bi)(x-a+bi) = (x-a)^2 + b^2.$$

Example: Roots $3, -\sqrt{2}, 2i$ with rational coefficients require $\sqrt{2}, -2i$, so a suitable polynomial is

$$\boxed{(x-3)(x^2-2)(x^2+4)}, \qquad \deg P = 5.$$

Note: The coefficient assumptions are essential; these pair rules do not apply to every polynomial with arbitrary coefficients.

7. Graph-to-factorisation reasoning

A graph gives approximate real roots and whether a curve crosses or touches. Use the exact rational roots in synthetic division; the remaining quadratic may have irrational or nonreal roots. Verify candidate roots algebraically.

8. Applications

When a volume, area, or average is modelled by polynomials, divide by known dimensions or quantities. Check that the resulting dimensions are positive and that all stated contextual constraints are consistent.



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