HL Lesson 2.6: Function Transformations

Student Worksheet – Detailed Markscheme at the End

Maths with Lemon

Mathematician: _____

Warm-up: coordinate transformations

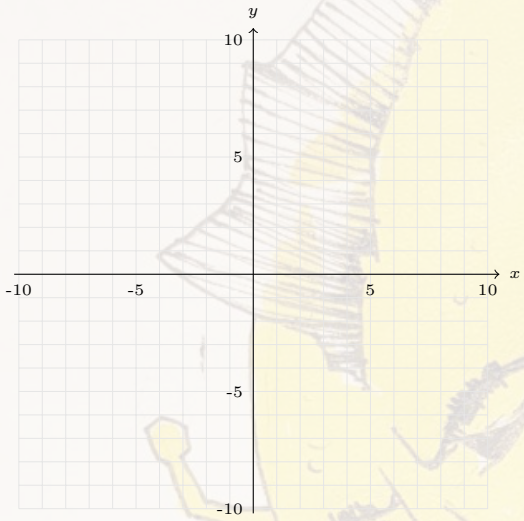
The piecewise linear graph f joins the points $(-6,-4),(-3,2),(0,0),(3,-2),(4,0),(6,1)$ in that order. Sketch each transformation, marking the images of these vertices.

(a) $f(x-2)+2$

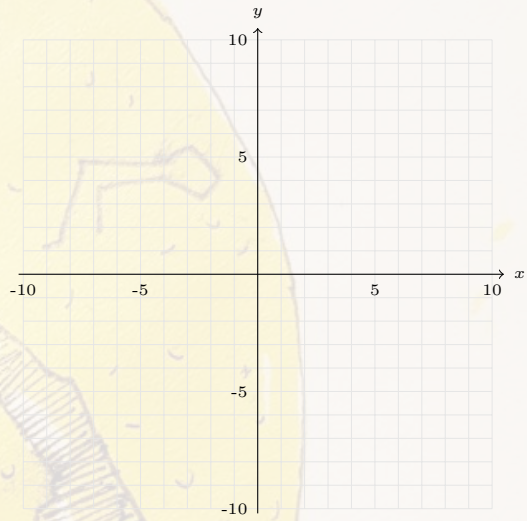
(b) $2f(x)-2$

(c) $f(2x+2)$

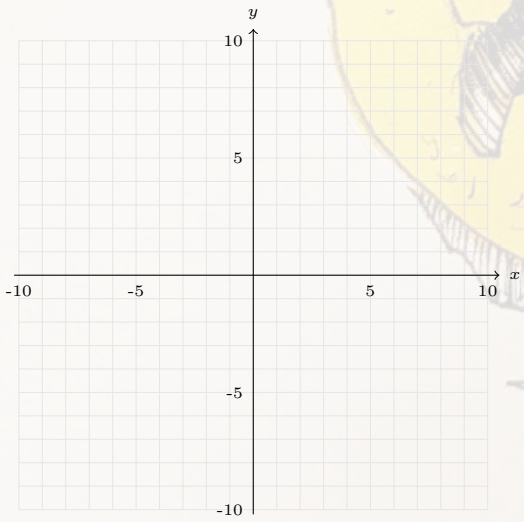
(d) $f(\frac{3}{2}x)$



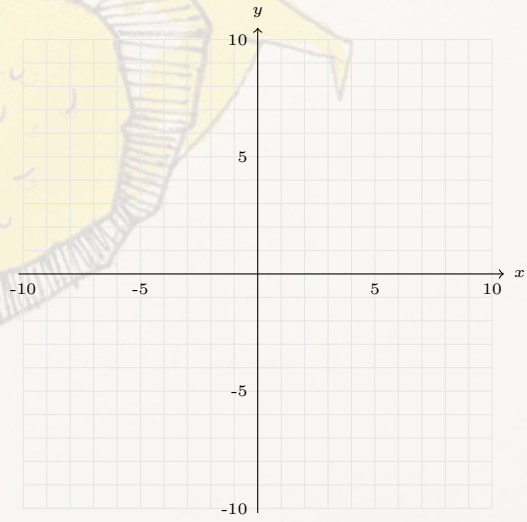
$f(x-2)+2$



$2f(x)-2$



$f(2x+2)$



$f(\frac{3}{2}x)$

Absolute-value transformations

Use the same piecewise linear graph f to sketch:

(a) $|f(x)|$

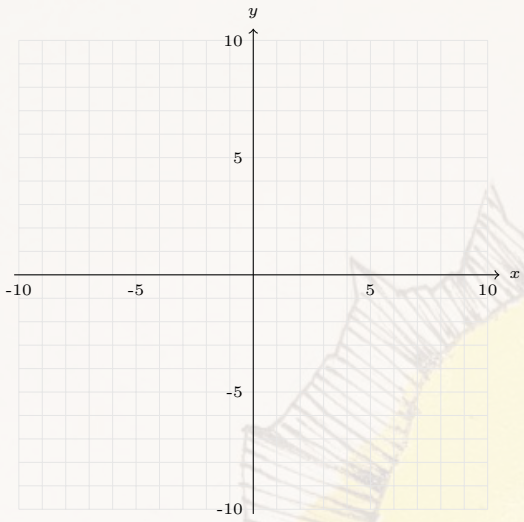
(b) $f(|x|)$

(c) $|f(x)| + 2$

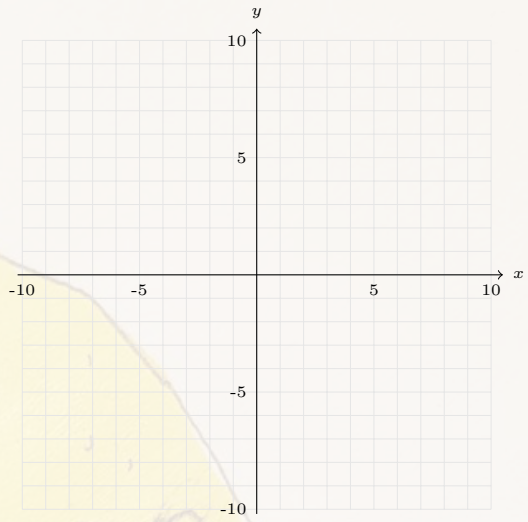
(d) $|f(x) + 2|$

(e) $f(|x| + 2)$

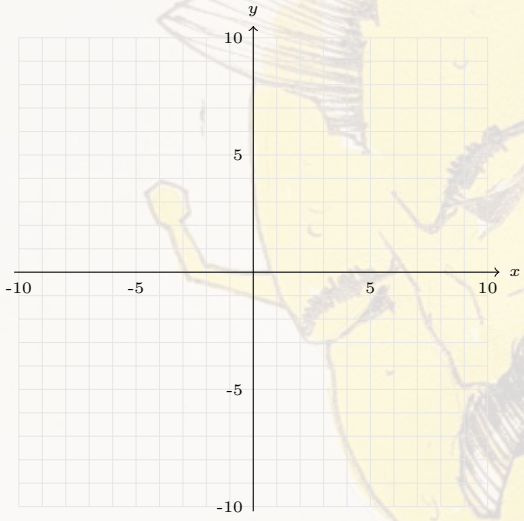
(f) $f(|x + 2|)$



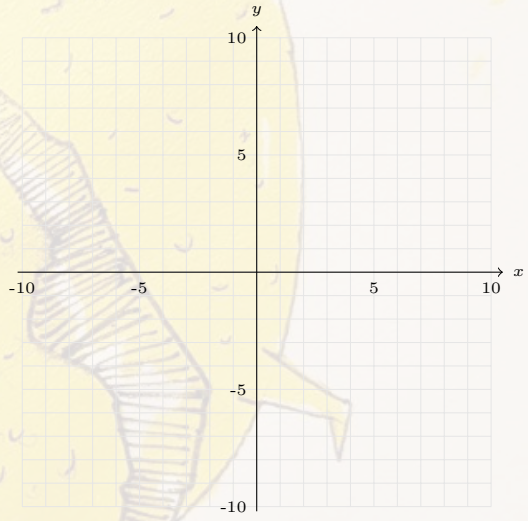
$|f(x)|$



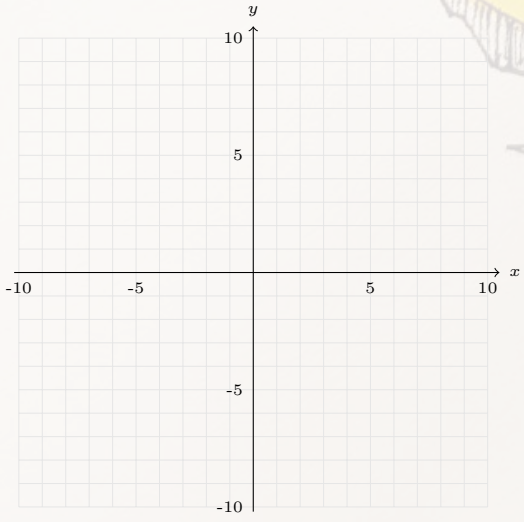
$f(|x|)$



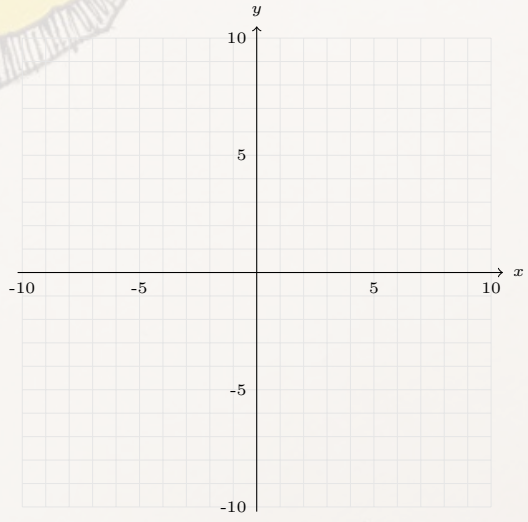
$|f(x)| + 2$



$|f(x) + 2|$



$f(|x| + 2)$



$f(|x + 2|)$

Cubic transformations

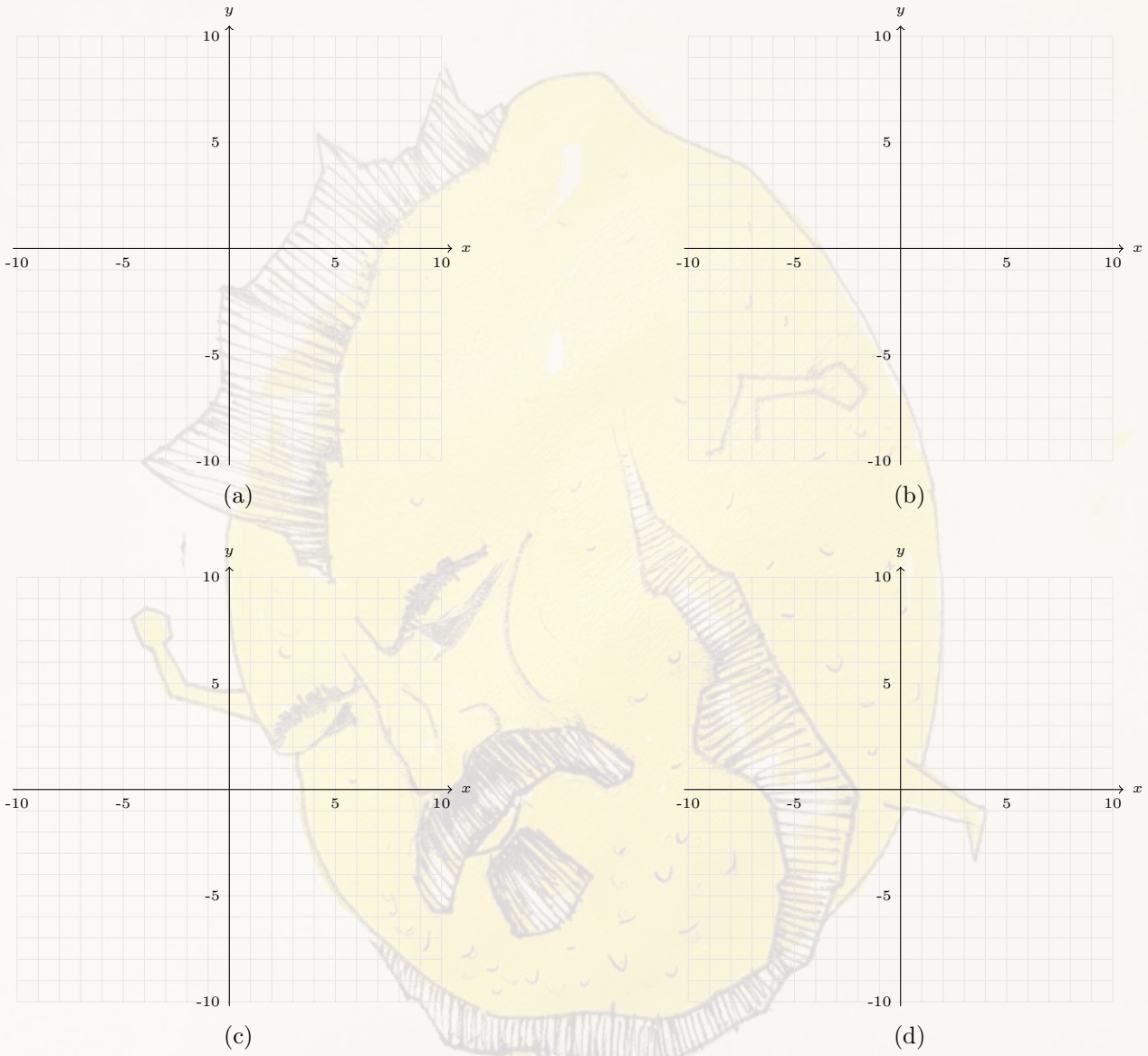
Let $c(x) = \frac{1}{2}(x-2)^3 - 3$. Sketch the following, showing intercepts and symmetry where relevant.

(a) $\left|\frac{1}{2}(x-2)^3\right| - 3$

(b) $\left|\frac{1}{2}(x-2)^3 - 3\right|$

(c) $\frac{1}{2}(|x|-2)^3 - 3$

(d) $\frac{1}{2}|x-2|^3 - 3$



(a) (b)

(c) (d)

Square-root transformations

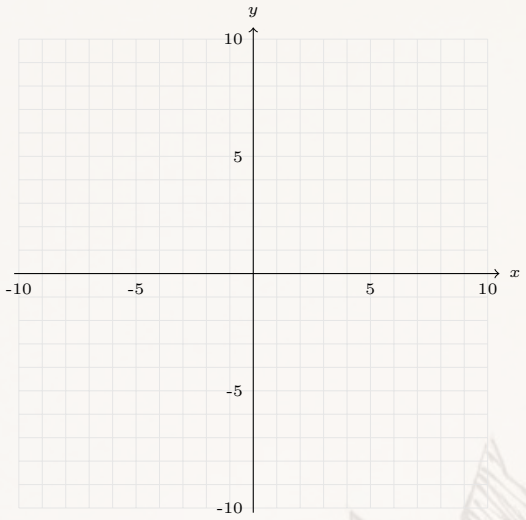
Let $r(x) = \sqrt{2(x+2)} - 4$. Sketch:

(a) $|\sqrt{2(x+2)}| - 4$

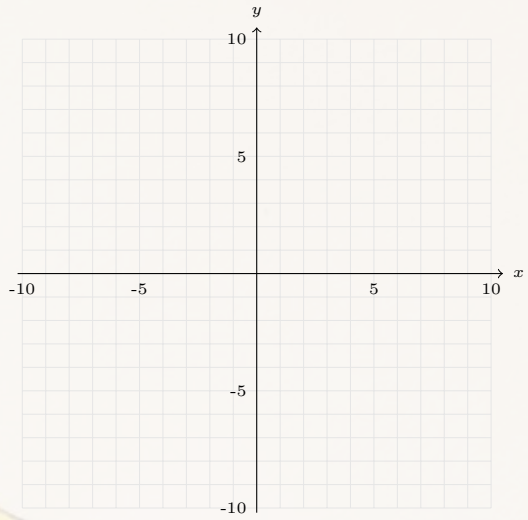
(b) $|\sqrt{2(x+2)} - 4|$

(c) $\sqrt{2(|x|+2)} - 4$

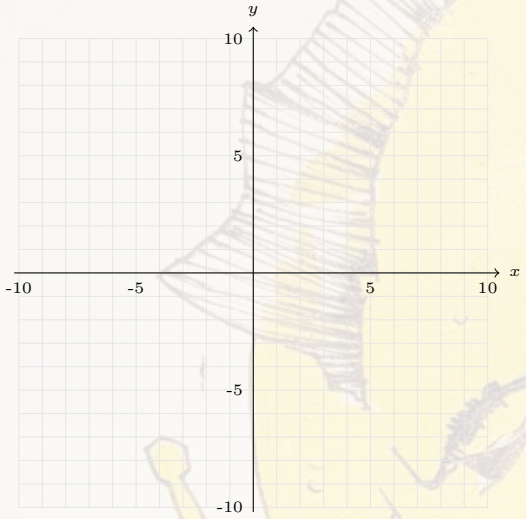
(d) $\sqrt{2|x+2|} - 4$



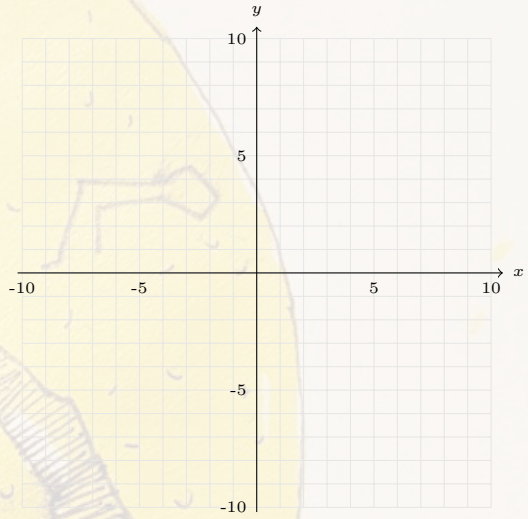
(a)



(b)



(c)



(d)

Logarithmic transformations

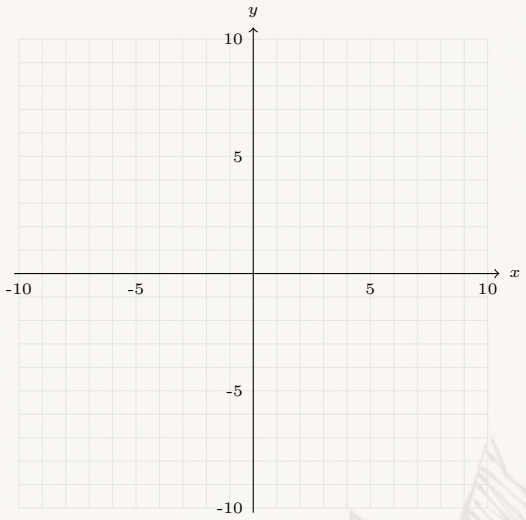
Let $\ell(x) = \log_2(x-1) - 2$. Sketch, identifying domains and asymptotes:

(a) $|\log_2(x-1)| - 2$

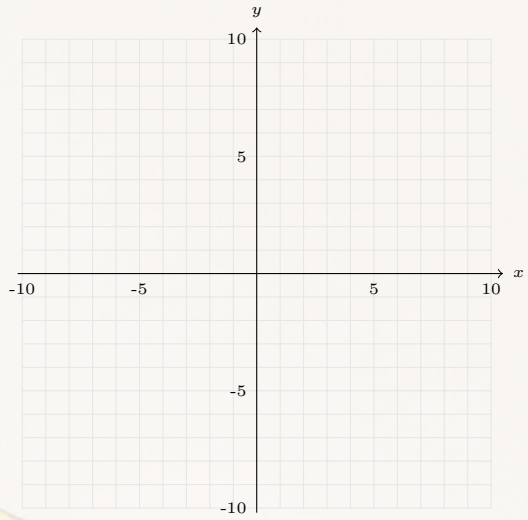
(b) $|\log_2(x-1) - 2|$

(c) $\log_2(|x|-1) - 2$

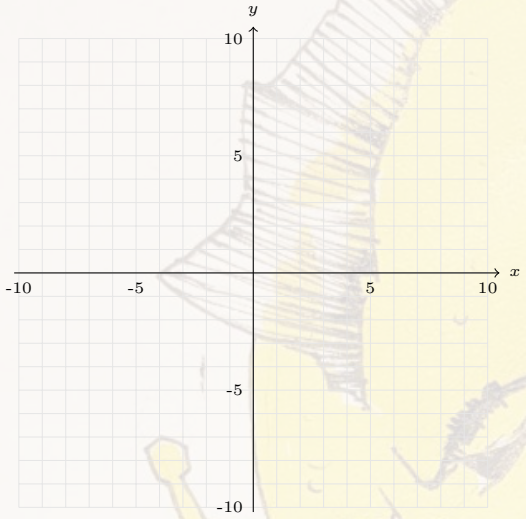
(d) $\log_2(|x-1|) - 2$



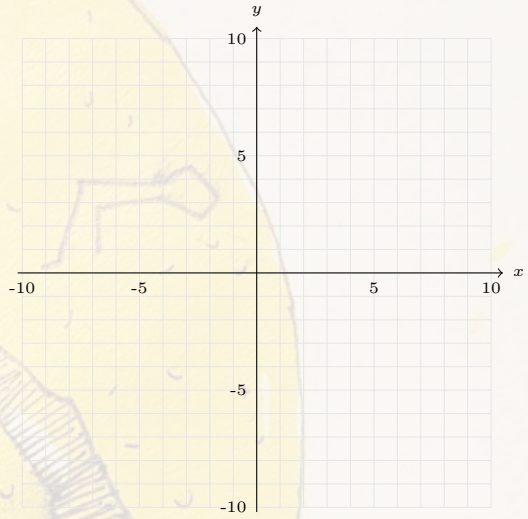
(a)



(b)



(c)



(d)

Reciprocal function transformations

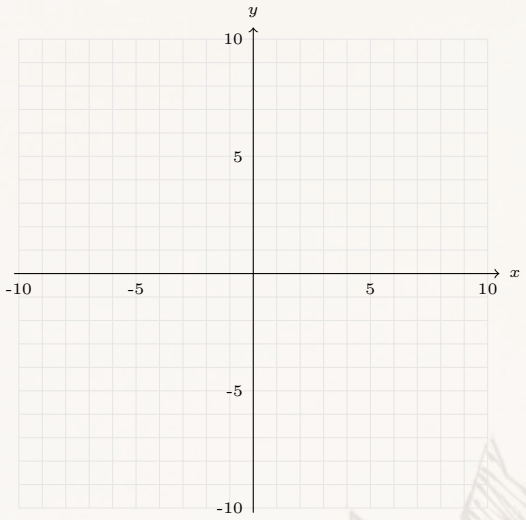
Let $q(x) = \frac{3}{x+2} + 3$. Sketch:

(a) $\left|\frac{3}{x+2}\right| + 3$

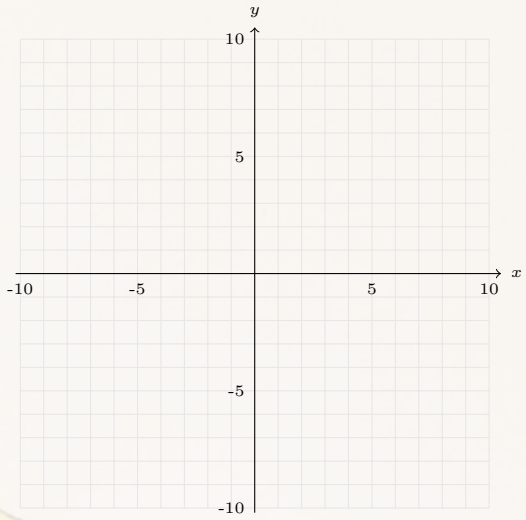
(b) $\left|\frac{3}{x+2} + 3\right|$

(c) $\frac{3}{|x|+2} + 3$

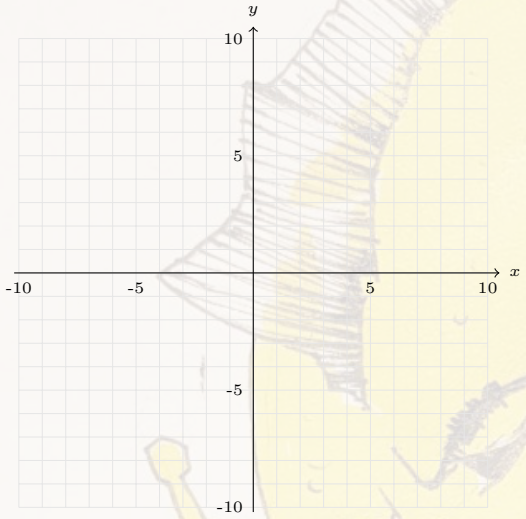
(d) $\frac{3}{|x+2|} + 3$



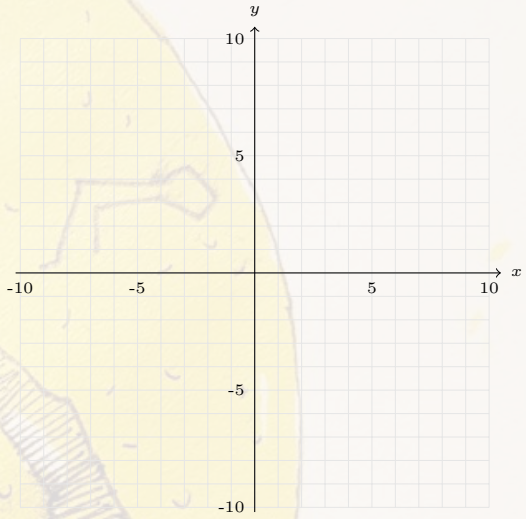
(a)



(b)



(c)



(d)

Square and reciprocal transformations

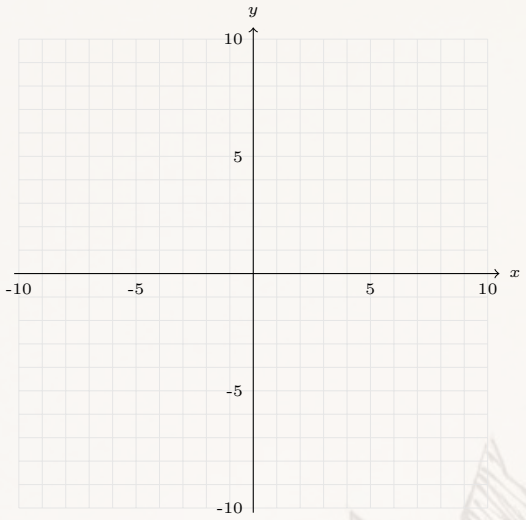
1. Sketch $[f(x)]^2$ and $1/f(x)$ for each of the following:

(a) $f(x) = \frac{3}{x+2} + 3$

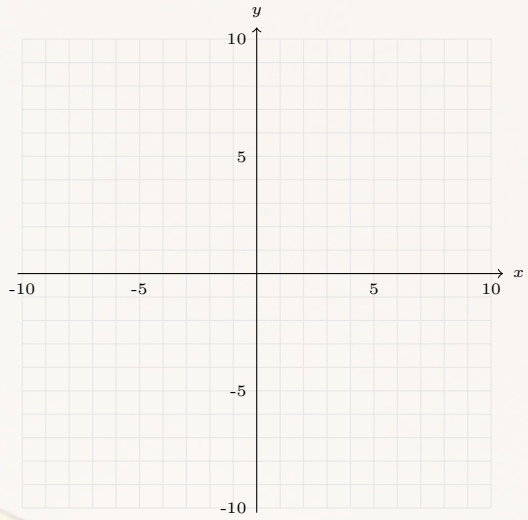
(b) $f(x) = \log_2(x-1) - 2$

(c) $f(x) = \frac{1}{2}(x-2)^3 - 3$

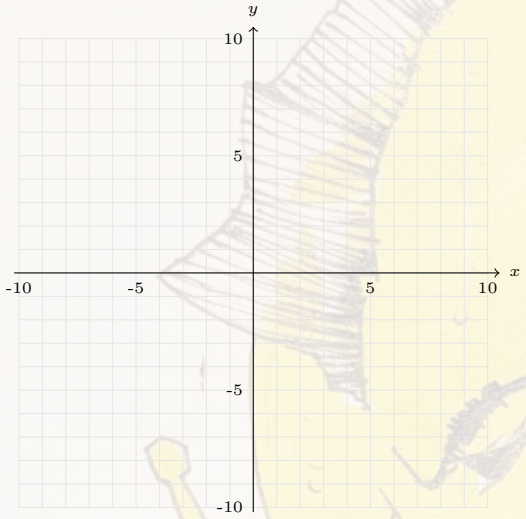
Label the zeros of f, any resulting vertical asymptotes, and horizontal asymptotes when present.



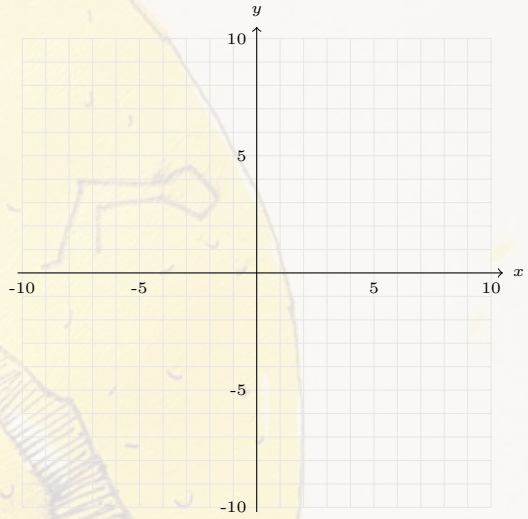
$[q(x)]^2$



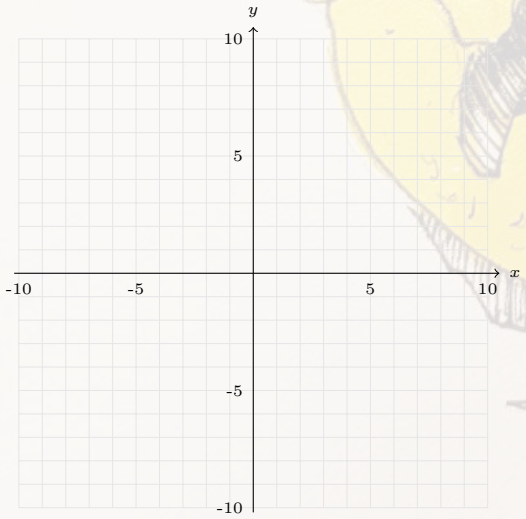
$1/q(x)$



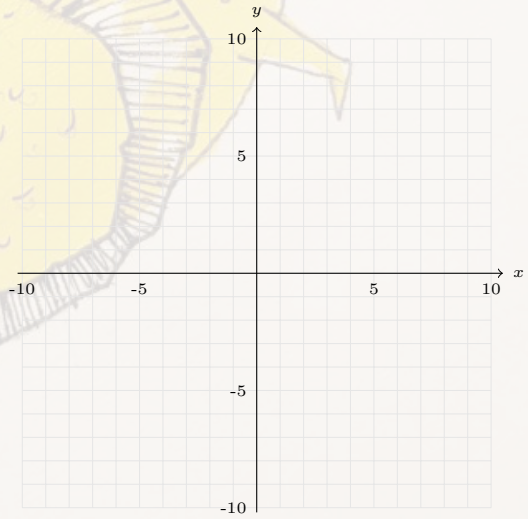
$[\ell(x)]^2$



$1/\ell(x)$



$[c(x)]^2$



$1/c(x)$

2. A graph of f has simple roots $-3, 0, 4$, and a local maximum at $(-2, 3)$ and minimum at $(2, -5)$. Describe the main features of $1/f(x)$ and $[f(x)]^2$.

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3. Another graph has roots $-6, 0, 3, 6$ and extrema $(-4, -8), (2, 2), (5, -3)$. Describe the transformed graphs $1/f(x)$ and $[f(x)]^2$.

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4. A graph f touches the x-axis at $x = \pm p$ and has a positive local maximum at $(0, c)$, $p, c > 0$. Sketch the reciprocal graph $1/[f(ax - b)]$ for $a, b > 0$, stating the locations of its vertical asymptotes.

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5. A reciprocal-type graph f has vertical asymptote $x = 0$, horizontal asymptote $y = c$, and a root at $x = a > 0$. Describe $[f(ax + b)]^2$ for $a, b > 0$, giving the transformed asymptotes and root.

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Trigonometric graphs

6. For $y = a\sin(bx + c)$, maxima occur at $(-7\pi/3, 4)$ and $(5\pi/3, 4)$, and a minimum at $(-\pi/3, -4)$. Find one suitable set of constants a, b, c.

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7. For $y = a + \sin(bx + c)$, maxima $(0, 0), (\pm 2\pi/3, 0)$ and minima $(\pm\pi/3, -2)$ are shown. Find a, b, c with $b > 0$.

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Challenges

8. State two transformations mapping $y = f(2x + 1)$ onto $y = f(3x)$.

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9. The graph of $y = \tan(3x - \pi/2)$ is translated left by $\pi/6$ and then reflected in the y-axis. Find the new equation.

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10. A vertical stretch by factor 5 of $y = 8^x$ equals the graph obtained by translating $y = 2^x$ right by c units and then stretching it horizontally by factor $\frac{1}{3}$. Find c.

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Detailed Analytical Markscheme

HL Lesson 2.6 – Function Transformations

Warm-up

A point (u, v) on the original graph maps to:

$$(u+2, v+2), \quad (u, 2v-2), \quad ((u-2)/2, v), \quad (2u/3, v).$$

Apply these rules to all six listed vertices and join them in order. For example, $(-3, 2)$ maps respectively to $(-1, 4), (-3, 2), (-5/2, 2), (-2, 2)$.

Absolute-value transformations

For each vertex (u, v), the output transformation gives $(u, |v|)$ for $|f(x)|$ and $(u, |v| + 2)$ for $|f(x)| + 2$. For $|f(x) + 2|$, the image is $(u, |v + 2|)$; new corners also occur where the original segments meet $y = -2$. For $f(|x|)$, retain the original part with $x \geq 0$ and reflect it across $x = 0$. For $f(|x| + 2)$, retain f for input $u \geq 2$, place $(u - 2, f(u))$ and $(-(u - 2), f(u))$. For $f(|x + 2|)$, retain f for $u \geq 0$, placing points at $(u - 2, f(u))$ and $(-u - 2, f(u))$. The axis of symmetry is $x = -2$.

Cubic transformations

(a) $\frac{1}{2}|x - 2|^3 - 3$; minimum $(2, -3)$, roots $x = 2 \pm \sqrt[3]{6}$, symmetric about $x = 2$. (b) $|\frac{1}{2}(x - 2)^3 - 3|$; zero at $x = 2 + \sqrt[3]{6}$; reflects negative values of the original cubic above the axis. (c) $\frac{1}{2}(|x| - 2)^3 - 3$; symmetric about $x = 0$, y-intercept -7, roots $x = \pm(2 + \sqrt[3]{6})$. (d) Same expression as (a), since $|\frac{1}{2}(x - 2)^3| = \frac{1}{2}|x - 2|^3$. The two graphs coincide.

Square-root transformations

(a) The original $\sqrt{2(x + 2)} - 4$ unchanged; domain $x \geq -2$, endpoint $(-2, -4)$, zero at $x = 6$. (b) $|\sqrt{2(x + 2)} - 4|$; domain $x \geq -2$, minimum $(6, 0)$, y-intercept 2. (c) $\sqrt{2(|x| + 2)} - 4$; domain $\mathbb{R}$, symmetric about y-axis, minimum $(0, -2)$, zeros $x = \pm 6$. (d) $\sqrt{2|x + 2|} - 4$; domain $\mathbb{R}$, symmetric about $x = -2$, minimum $(-2, -4)$, zeros $x = -10, 6$.

Logarithmic transformations

(a) $|\log_2(x - 1)| - 2$: domain $x > 1$, vertical asymptote $x = 1$; minimum $(2, -2)$, roots $x = 5/4, 5$. (b) $|\log_2(x - 1) - 2|$: domain $x > 1$, asymptote $x = 1$; minimum $(5, 0)$. (c) $\log_2(|x| - 1) - 2$: domain $x < -1$ or $x > 1$, asymptotes $x = -1, 1$; roots $x = \pm 5$; symmetric about y-axis. (d) $\log_2(|x - 1|) - 2$: domain $x \neq 1$, asymptote $x = 1$; roots $x = -3, 5$; symmetric about $x = 1$.

Reciprocal-function transformations

(a) $3/|x + 2| + 3$: domain $x \neq -2$; vertical asymptote $x = -2$, horizontal asymptote $y = 3$; symmetric about $x = -2$. (b) $|3/(x + 2) + 3|$: domain $x \neq -2$; zero at $x = -3$; vertical asymptote $x = -2$, horizontal asymptote $y = 3$. (c) $3/(|x| + 2) + 3$: domain all reals; symmetric about y-axis, maximum $(0, 9/2)$, horizontal asymptote $y = 3$. (d) $3/|x + 2| + 3$, identical to (a).

Square and reciprocal transformations

1(a). $q(x) = 3(x + 3)/(x + 2)$. Zero $x = -3$; pole $x = -2$; horizontal asymptote $y = 3$. $[q(x)]^2$ has a touching root at -3, vertical asymptote -2 and horizontal asymptote $y = 9$. $1/q(x) = (x+2)/[3(x+3)]$, with $x \neq -2, -3$; hole at $(-2, 0)$, vertical asymptote $x = -3$, horizontal asymptote $y = 1/3$. **1(b).** $\ell(x) = \log_2(x - 1) - 2$ has zero $x = 5$, domain $x > 1$, asymptote $x = 1$. $[\ell(x)]^2$ touches at $(5, 0)$ and tends to $+\infty$ as $x \to 1^+$ and as $x \to +\infty$. $1/\ell(x)$ has vertical asymptote $x = 5$; $x = 1$ remains excluded, with right-hand limit 0^-; as $x \to \infty$, it tends to 0^+. **1(c).** $c(x) = \frac{1}{2}(x - 2)^3 - 3$ has zero $r = 2 + \sqrt[3]{6}$. $[c(x)]^2$ touches the axis at r; $1/c(x)$ has vertical asymptote $x = r$ and horizontal asymptote $y = 0$. **2.** $1/f$ has vertical asymptotes at $-3, 0, 4$ (assuming simple roots) and points $(-2, 1/3)$ and $(2, -1/5)$. $[f]^2$ touches the axis at the three roots, with $(-2, 9)$ and $(2, 25)$ as transformed stationary points (their local types may change according to sign). **3.** $1/f$ has vertical asymptotes at $-6, 0, 3, 6$ if the roots are simple; stationary values at $(-4, -1/8), (2, 1/2), (5, -1/3)$. $[f]^2$ has roots at the four zeros and stationary points $(-4, 64), (2, 4), (5, 9)$. **4.** Zeros of $f(ax - b)$ occur at $x = (b - p)/a$ and $(b + p)/a$, giving vertical asymptotes for the reciprocal (provided no cancellation); $x = b/a$ corresponds to the positive maximum c of f, so the reciprocal takes value $1/c$ there. **5.** The transformed vertical asymptote

is $x = -b/a$; the horizontal asymptote becomes $y = c^2$; the transformed root is $x = (a-b)/a = 1 - b/a$. Note that the a in $f(ax+b)$ is also used for the original root as stated in the source.

Trigonometric graphs

6. Amplitude $|a| = 4$. Period $5\pi/3 - (-7\pi/3) = 4\pi$, so $b = 1/2$. Choosing $a = 4$, a maximum at $x = 5\pi/3$ requires $\frac{1}{2}(5\pi/3) + c = \pi/2 + 2k\pi$, hence $c = -\pi/3 \pmod{2\pi}$. Thus $\boxed{a = 4, b = \frac{1}{2}, c = -\frac{\pi}{3}}$. **7.** Midline $a = -1$, amplitude 1, period $2\pi/3$, so $b = 3$. A maximum at $x = 0$ requires $\sin c = 1$, hence $c = \pi/2 \pmod{2\pi}$. Therefore $\boxed{a = -1, b = 3, c = \pi/2}$.

Challenges

8. Translate $f(2x+1)$ left by $1/2$ to obtain $f(2x+2)$, then horizontally stretch by factor $2/3$ to obtain $f(3x)$. **9.** After translating left $\pi/6$: $\tan(3(x+\pi/6) - \pi/2) = \tan(3x)$. Reflect in the y-axis: $\boxed{y = \tan(-3x) = -\tan(3x)}$. **10.** A horizontal stretch by $1/3$ of 2^{x-c} produces 2^{3x-c}. Set $2^{3x-c} = 5 \cdot 8^x = 5 \cdot 2^{3x}$; hence $2^{-c} = 5$, so $\boxed{c = -\log_2 5}$. The negative value means that the stated translation "right by c" is in fact left by $\log_2 5$ units.





Maths with Lemon

Mathematics made clear.