HL Lesson 2.6: Function Transformations

Theory and Worked Examples – AHL 2.16

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1. The transformation language

For a point (u, v) on $y = f(x)$, the following coordinate mappings are useful:

New graph	Mapping / effect
$y = f(x-h)+k$	$(u,v) \mapsto (u+h, v+k)$: right h, up k.
$y = af(x)+k$	$(u,v) \mapsto (u, av+k)$: vertical scaling and translation.
$y = f(ax+b)$	$(u,v) \mapsto ((u-b)/a, v)$ for $a \neq 0$.
$y = \lvert f(x)\rvert$	$(u,v) \mapsto (u, \lvert v\rvert)$: negative outputs reflected above x-axis.
$y = f(\lvert x\rvert)$	Retain the right-hand side ($x \geq 0$), then reflect it in the y-axis.
$y = [f(x)]^2$	$(u,v) \mapsto (u, v^2)$; outputs become nonnegative.
$y = 1/f(x)$	$(u,v) \mapsto (u, 1/v)$, only when $v \neq 0$.

2. Worked example: translations and stretches

Let $f(x) = x^2$. Sketch $g(x) = f(2x+2) - 3 = (2x+2)^2 - 3$.

$$2x + 2 = 2(x + 1).$$

Step 1: Horizontally compress by factor $\frac{1}{2}$.
Step 2: Translate left 1 and down 3.
Step 3: The vertex is $(-1, -3)$; axis $x = -1$.
Step 4: The zeros solve $(2x+2)^2 = 3$, hence $x = -1 \pm \frac{\sqrt{3}}{2}$.

3. Absolute-value transformations: outside vs inside

$$\boxed{y = |f(x)|} \qquad \boxed{y = f(|x|)}$$

For $|f(x)|$, the *negative part of the graph* is reflected in the x-axis. For $f(|x|)$, *discard the left-hand part* and mirror the right-hand part.

$$|f(x)| + k \neq |f(x) + k| \quad \text{in general.}$$

Worked example: $f(x) = x - 2$.

$$|f(x)| = |x-2| = \begin{cases} 2-x, & x < 2, \\ x-2, & x \geq 2. \end{cases}$$

But $f(|x|) = |x| - 2$ has vertex $(0, -2)$, whereas $|f(x)|$ has vertex $(2, 0)$.

4. Absolute values and cubic functions

For $f(x) = \frac{1}{2}(x-2)^3 - 3$:

- $|\frac{1}{2}(x-2)^3| - 3$: reflect the cubic's negative *inner output* before shifting down 3.
- $|\frac{1}{2}(x-2)^3 - 3|$: reflect all negative values of the *whole function*.
- $\frac{1}{2}(|x|-2)^3 - 3$: mirror the original graph's $x \geq 0$ half.
- $\frac{1}{2}|x-2|^3 - 3$: reflect the part of the cubic centred at $x = 2$ to form a symmetric graph about $x = 2$.

5. Domain matters: square roots and logarithms

For $\sqrt{2(x+2)} - 4$, the domain is $x \geq -2$. The endpoint is $(-2, -4)$.

$$\left|\sqrt{2(x+2)}\right| - 4 = \sqrt{2(x+2)} - 4$$

because a square root is already nonnegative. However $\left|\sqrt{2(x+2)}-4\right|$ reflects the part below the x-axis; it reaches zero at $x=6$. For $\log_2(x-1)-2$, the domain is $x>1$ and the vertical asymptote is $x=1$. The absolute-value output $|\log_2(x-1)-2|$ keeps the same domain and asymptote, but its graph stays above the axis. The inside transformation $\log_2(|x|-1)-2$ has domain $x<-1$ or $x>1$, with asymptotes $x=\pm 1$.

6. Reciprocal graphs

For $g(x)=1/f(x)$:

- Zeros of f are excluded from the domain of g; simple zeros typically produce vertical asymptotes.
- If $f(x)=1$ or $f(x)=-1$, the reciprocal has the same value.
- $f(x)>0$ implies $g(x)>0$, and similarly for negative values.
- If $f(x)\to\pm\infty$, then $g(x)\to 0$.
- A zero of even multiplicity may give the same infinity sign on both sides.

Worked example: $f(x)=x^2-4$ gives $g(x)=1/(x^2-4)$. The domain excludes $x=\pm 2$, both vertical asymptotes; horizontal asymptote $y=0$; $g(0)=-1/4$.

7. Squaring a function

For $g(x)=[f(x)]^2$, all outputs are nonnegative. Roots of f remain roots and become touching points when they are simple and the graph is smooth. If $0<|f(x)|<1$, then $[f(x)]^2<|f(x)|$; if $|f(x)|>1$, squaring increases the magnitude. **Example:** $f(x)=x-3$ gives $g(x)=(x-3)^2$, with a minimum at $(3,0)$.

8. Composition and order of transformations

To transform $f(2x+1)$ into $f(3x)$, first translate the graph left by $\frac{1}{2}$ to obtain $f(2x+2)$, then horizontally stretch by factor $\frac{2}{3}$ to obtain $f(3x)$. **Check:** a horizontal stretch by factor s maps $g(x)$ to $g(x/s)$.

9. Common errors

- $f(2x+2)$ means a compression by $\frac{1}{2}$ and a translation left by 1, not left by 2.
- $f(|x-2|)$ is symmetric about $x=2$, whereas $f(|x|-2)$ is symmetric about the y-axis.
- $|f(x)|+2$ is not the same as $|f(x)+2|$.
- Squaring and reciprocating outputs change the range and may change the asymptotes; always re-check the domain.



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Mathematics made clear.