HL Lesson 2.8: Function Inverses

Student Worksheet – AHL 2.14

Maths with Lemon

Mathematician: _____

1. Inverse function domain restrictions

Consider $f(x) = x^2 + 1$.

(a) Explain why f does not have an inverse function on $\mathbb{R}$.

(b) Restrict the domain so that f has an inverse.

(c) Find the inverse for your chosen restriction and state its domain.

2. Self-inverse functions

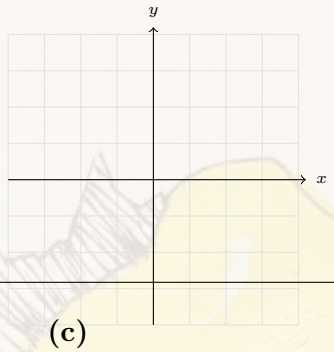
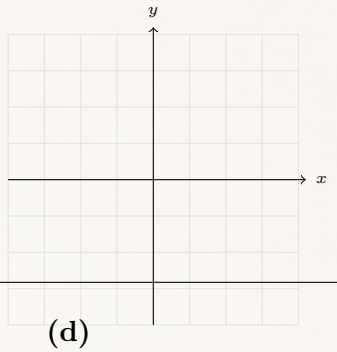
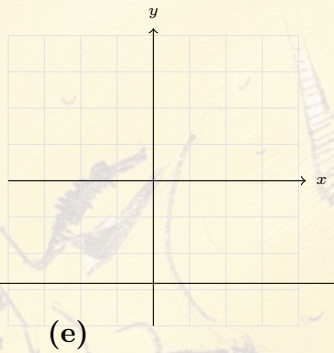
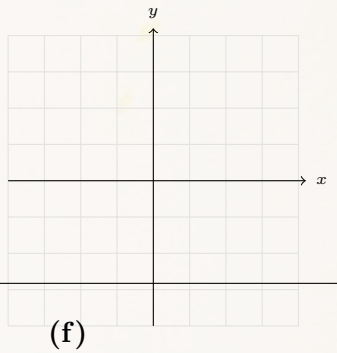
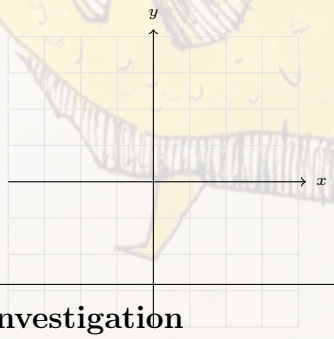
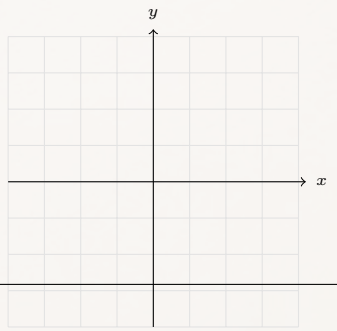
(a) Find the inverse of $f(x) = \dfrac{5}{x-3} + 3$, $x \neq 3$.

(b) Explain the relationship between the graph and the line $y = x$.

(c) For $f(x) = \dfrac{k}{x-1} + 1$, $x > 1$, show it is self-inverse for admissible k. State the condition on k.

3. Restricting domains and sketching inverses

For each function, fill in the restriction boundary, find f^{-1}, state $\text{Dom}(f^{-1})$, and sketch f and f^{-1} on the same

Function A	Function B
(a) $y = x^2 - 5,\ x \geq ___$ Domain restriction: _____ Inverse: _____ Inverse domain: _____ 	**(b)** $y = (x+3)^2 + 1,\ x \geq ___$ Domain restriction: _____ Inverse: _____ Inverse domain: _____ 
(c) $y = 2(x-1)^2,\ x \geq ___$ Domain restriction: _____ Inverse: _____ Inverse domain: _____ 	**(d)** $y = -3x^2 + 2,\ x \leq ___$ Domain restriction: _____ Inverse: _____ Inverse domain: _____ 
(e) $y = (x-4)^4 - 2,\ x \leq ___$ Domain restriction: _____ Inverse: _____ Inverse domain: _____ 	**(f)** $y = -x^4 + 4,\ x \leq ___$ Domain restriction: _____ Inverse: _____ Inverse domain: _____ 

axes, showing $y = x$.

4. Odd and even functions: investigation

For x^2, x, $x + \frac{1}{x}$, $x^2 - \frac{1}{x^2}$, $2x^3 + 3x$, and $-x^4 + 2x^2$:

(a) Determine whether $f(-x) = f(x)$.

(b) Determine whether $f(-x) = -f(x)$.

(c) Classify each as even, odd, or neither and justify algebraically.

(d) Explain whether a function can be neither even nor odd.

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5. Identify parity without a calculator

Classify each as odd, even or neither and show your reasoning.

(a) $2x + 3$

(b) $x^2 + \dfrac{3}{x}$

(c) $2x^3 - x$

(d) $\sin x$

(e) $\sec x$

(f) $2x - \dfrac{x^3}{3}$

(g) $\sqrt{x+1}$

(h) $\cos(x^2)$

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6. Multiple choice: Which is odd?

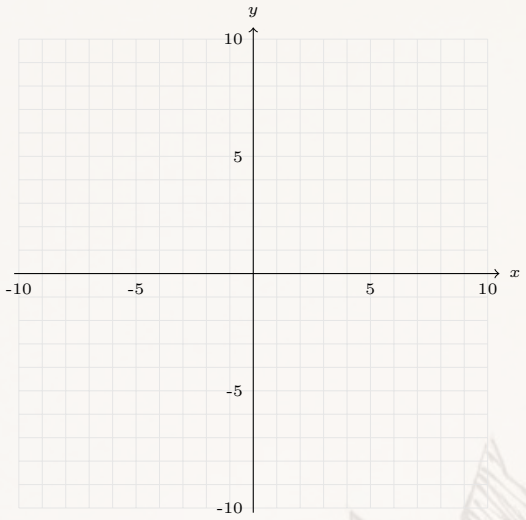
A $x - \dfrac{x^3}{2}$ B $3 + x$ C $\sqrt{1+x^2}$ D $x^2 - 3x + 1$.

Which is even?

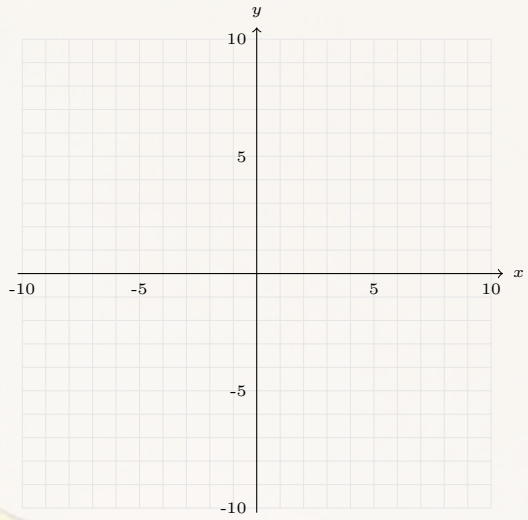
A $\sqrt{\dfrac{1}{x^2} + x^2}$ B $3x - 2$ C $x^3 - \dfrac{x}{2}$ D $x^4 - 3x + 1$.

7. Identify graphs by symmetry

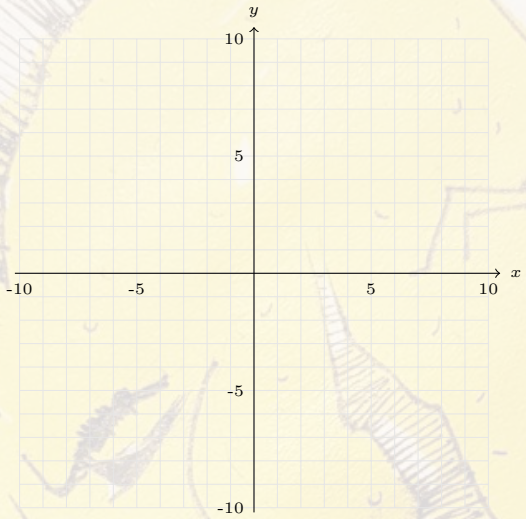
Sketch a graph that is (a) even, (b) odd, and (c) neither, and label the relevant symmetry.



Even



Odd



Neither

8. Additional practice problems

Problem 1. Which of the following can be a domain of $f(x) = x^2 + 4x$ if f has an inverse?

(a) $x \leq -2$ (b) $x \leq -1$ (c) $x \geq -3$ (d) $x \leq 4$.

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Problem 2. Let $f(x) = x^3 + 6x^2 + 9x - 2$ on $-5 \leq x \leq 1$.

(a) Find the largest possible domain(s) of the form $a \leq x \leq b$ on which f has an inverse.

(b) For each such interval, find $\text{Dom}(f^{-1})$.

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Problem 3. f is even and g is odd.

(a) Prove that $h(x) = f(x)/g(x)$ is odd, wherever defined.

(b) With proof, determine whether the sum of an even and an odd function is even, odd or neither in general.

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Problem 4. For any function f on a domain symmetric about zero:

(a) Prove $f(x) + f(-x)$ is even.

(b) Prove $f(x) - f(-x)$ is odd.

(c) Express f as the sum of an even and an odd function.

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Problem 5. Find c for which $f(x) = \dfrac{3-2x}{x+c}$ is self-inverse.

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Detailed Analytical Markscheme

HL Lesson 2.8 – Function Inverses

1. Quadratic domain restriction

(a) $f(-1) = f(1) = 2$, so it fails the horizontal line test. (b) Either $x \geq 0$ or $x \leq 0$ works. (c) For $x \geq 0$:

$$x = y^2 + 1 \Rightarrow y = \sqrt{x-1} \Rightarrow \boxed{f^{-1}(x) = \sqrt{x-1},\ x \geq 1}.$$

For $x \leq 0$: $f^{-1}(x) = -\sqrt{x-1}$, $x \geq 1$.

2. Self-inverse functions

(a) $y = 5/(x-3) + 3$ implies $(y-3)(x-3) = 5$, so

$$\boxed{f^{-1}(x) = 3 + \frac{5}{x-3},\ x \neq 3}.$$

(b) The graph is invariant under reflection in $y = x$; points (a, b) and (b, a) both belong to it. (c) For $k > 0$, f maps $(1, \infty)$ onto itself, and

$$f(f(x)) = 1 + \frac{k}{(1 + k/(x-1)) - 1} = 1 + \frac{k}{k/(x-1)} = x.$$

Thus $f^{-1} = f$. For $k \leq 0$, the stated domain is not mapped onto itself (or f is constant when $k = 0$).

3. Restricted inverse equations and graph features

Each inverse is the reflection of the restricted original graph in $y = x$.

(a) $x \geq 0$: $y = x^2 - 5 \Rightarrow x = y^2 - 5$, $y \geq 0$, hence $\boxed{f^{-1}(x) = \sqrt{x+5},\ x \geq -5}$. Endpoint swaps $(0, -5) \leftrightarrow (-5, 0)$.

(b) $x \geq -3$: $\boxed{f^{-1}(x) = -3 + \sqrt{x-1},\ x \geq 1}$. Endpoint $(-3, 1) \leftrightarrow (1, -3)$.

(c) $x \geq 1$: $\boxed{f^{-1}(x) = 1 + \sqrt{x/2},\ x \geq 0}$. Endpoint $(1, 0) \leftrightarrow (0, 1)$.

(d) $x \leq 0$: $\boxed{f^{-1}(x) = -\sqrt{(2-x)/3},\ x \leq 2}$. Endpoint $(0, 2) \leftrightarrow (2, 0)$.

(e) $x \leq 4$: $\boxed{f^{-1}(x) = 4 - \sqrt[4]{x+2},\ x \geq -2}$. Endpoint $(4, -2) \leftrightarrow (-2, 4)$.

(f) $x \leq 0$: $\boxed{f^{-1}(x) = -\sqrt[4]{4-x},\ x \leq 4}$. Endpoint $(0, 4) \leftrightarrow (4, 0)$.

4. Parity investigation

Function	$f(-x)$	Classification
x^2	x^2	Even
x	$-x$	Odd
$x + 1/x$	$-x - 1/x$	Odd
$x^2 - 1/x^2$	$x^2 - 1/x^2$	Even
$2x^3 + 3x$	$-2x^3 - 3x$	Odd
$-x^4 + 2x^2$	$-x^4 + 2x^2$	Even

A function can be neither (e.g. $f(x) = x + 1$); yes, substitution $-x$ verifies parity algebraically.

5. Classification with reasoning

(a) Neither: $f(-x) = -2x + 3$. (b) Neither: $f(-x) = x^2 - 3/x$. (c) Odd: $f(-x) = -2x^3 + x = -f(x)$. (d) Odd: $\sin(-x) = -\sin x$. (e) Even: $\sec(-x) = 1/\cos(-x) = \sec x$ on its symmetric domain. (f) Odd:

$f(-x) = -2x + x^3/3 = -f(x)$. (g) Neither: $\sqrt{x+1}$ has natural domain $[-1, \infty)$, not symmetric about zero. (h) Even: $\cos((-x)^2) = \cos(x^2)$.

6. Multiple choice

Odd: **A**, since $f(-x) = -x + x^3/2 = -f(x)$. Even: **A**, since the radicand $1/x^2 + x^2$ is unchanged by $x \mapsto -x$ (domain $x \neq 0$).

7. Graph symmetry

Even: symmetry about the y-axis, e.g. $y = x^2$. Odd: rotational symmetry about origin, e.g. $y = x^3$. Neither: e.g. $y = x^2 + x$.

8. Additional practice

Problem 1. $f(x) = x^2 + 4x = (x+2)^2 - 4$. The vertex is at $x = -2$. Only **(a)** $x \leq -2$ lies entirely on one monotonic branch.

Problem 2. $f'(x) = 3(x+3)(x+1)$; turning points $x = -3, -1$. The three maximal monotonic subintervals within $[-5, 1]$ are $[-5, -3]$, $[-3, -1]$, and $[-1, 1]$; all have length 2. There is no unique longest interval.

$$f(-5) = -22, \quad f(-3) = -2, \quad f(-1) = -6, \quad f(1) = 14.$$

Hence the inverse domains are $[-22, -2]$, $[-6, -2]$, and $[-6, 14]$, respectively. A unique answer requires a further restriction not stated in the source.

Problem 3. (a) If f even and g odd, $h(-x) = f(-x)/g(-x) = f(x)/[-g(x)] = -h(x)$, provided $g(x) \neq 0$ and the domain is symmetric. (b) For $s = f + g$, $s(-x) = f(x) - g(x)$, which is generally neither $s(x)$ nor $-s(x)$. Example $x^2 + x$ is neither. If one summand is identically zero, exceptions are possible.

Problem 4. (a) $E(-x) = f(-x) + f(x) = E(x)$. (b) $O(-x) = f(-x) - f(x) = -O(x)$. (c) Write $\boxed{f(x) = \dfrac{f(x) + f(-x)}{2} + \dfrac{f(x) - f(-x)}{2}}$, even plus odd.

Problem 5. Swap x, y in $y = (3 - 2x)/(x + c)$:

$$yx + cy = 3 - 2x \Rightarrow x = \frac{3 - cy}{y + 2} \Rightarrow f^{-1}(x) = \frac{3 - cx}{x + 2}.$$

For a nonconstant self-inverse function, $f^{-1} = f$; this requires $\boxed{c = 2}$. Check $f(f(x)) = x$ for $x \neq -2$. The degenerate value $c = -3/2$ gives a constant function and has no inverse.



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Mathematics made clear.