HL Lesson 2.8: Function Inverses

Theory and Guided Worked Examples – AHL 2.14

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1. One-to-one functions and inverses

A function has an inverse *function* precisely when it is one-to-one on its domain: distinct inputs give distinct outputs. A graph passes the **horizontal line test** when it is one-to-one.

$$f(a) = f(b) \Longrightarrow a = b.$$

The inverse reverses input and output. If (a, b) is on $y = f(x)$, then (b, a) is on $y = f^{-1}(x)$. The two graphs are reflections in $y = x$.

$$\boxed{\mathrm{Dom}(f^{-1}) = \mathrm{Ran}(f)}, \qquad \boxed{\mathrm{Ran}(f^{-1}) = \mathrm{Dom}(f)}.$$

Method: write $y = f(x)$; interchange x, y; solve for y; state domain and range.

2. Restricting the domain of a quadratic

Worked example: $f(x) = x^2 + 1$ is not one-to-one on $\mathbb{R}$, since $f(-2) = f(2) = 5$. Choose $x \geq 0$ to make it one-to-one.

$$y = x^2 + 1 \implies x = y^2 + 1 \implies y = \sqrt{x - 1}.$$

Thus

$$\boxed{f^{-1}(x) = \sqrt{x - 1},\ x \geq 1}; \quad \mathrm{Ran}(f^{-1}) = [0, \infty).$$

If instead $\mathrm{Dom}(f) = (-\infty, 0]$, the inverse is $-\sqrt{x - 1}$, $x \geq 1$.

3. Guided pair: quadratic and quartic restrictions

Worked Example	You Try
Solve: $f(x) = (x + 3)^2 + 1$, $x \geq -3$. $y = (x + 3)^2 + 1$; exchange x, y: $x = (y + 3)^2 + 1$. $y + 3 = \sqrt{x - 1}$ (positive branch). $\boxed{f^{-1}(x) = -3 + \sqrt{x - 1},\ x \geq 1}$.	**Your turn:** $g(x) = (x - 4)^4 - 2$, $x \leq 4$. Find g^{-1} and its domain. ....

4. Self-inverse functions

A function is **self-inverse** if $f^{-1} = f$, equivalently $f(f(x)) = x$ throughout the domain, and f maps its domain onto itself. **Worked example:** $f(x) = 3 + \dfrac{5}{x - 3}$, $x \neq 3$.

$$y - 3 = \frac{5}{x - 3} \implies (x - 3)(y - 3) = 5 \implies x = 3 + \frac{5}{y - 3}.$$

Hence $f^{-1}(x) = f(x)$; the graph is symmetric about $y = x$. Note the two branches are separated by $x = 3$ and $y = 3$. **Parameter example:** $f(x) = 1 + \dfrac{k}{x - 1}$, $x > 1$. For the function to map $(1, \infty)$ to itself and be self-inverse, $k > 0$. Then

$$f(f(x)) = 1 + \frac{k}{k/(x - 1)} = x.$$

If $k = 0$, the map is constant; if $k < 0$, outputs are below 1, so it does not map the stated domain to itself.

5. Even and odd functions

For a domain symmetric about 0:

$$\boxed{\text{Even: } f(-x) = f(x)} \qquad \boxed{\text{Odd: } f(-x) = -f(x)}.$$

Even functions are symmetric about the y-axis; odd functions are symmetric about the origin (a half-turn).

Function	Classification and reason
x^2	Even: $(-x)^2 = x^2$.
$x + 1/x$ $(x \neq 0)$	Odd: $-x - 1/x = -f(x)$.
$x^2 - 1/x^2$ $(x \neq 0)$	Even: both terms unchanged.
$2x^3 + 3x$	Odd: both terms change sign.
$-x^4 + 2x^2$	Even: both terms unchanged.
$2x + 3$	Neither: $-2x + 3$ equals neither $f(x)$ nor $-f(x)$.

Caution: The domain must also be symmetric about zero. For example $\sqrt{x+1}$ is neither even nor odd on its natural domain.

6. Proofs with parity

Suppose f is even and g is odd, and $g(x) \neq 0$ on a symmetric domain. For $h(x) = f(x)/g(x)$,

$$h(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -h(x),$$

so h is odd. **Decomposition theorem:** For any f on a domain symmetric about zero,

$$\boxed{f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}}.$$

7. Monotonic intervals and inverses

For $f(x) = x^3 + 6x^2 + 9x - 2$,

$$f'(x) = 3x^2 + 12x + 9 = 3(x+1)(x+3).$$

The function increases for $x < -3$, decreases for $-3 < x < -1$, and increases for $x > -1$. Hence on $[-5, 1]$ it can be restricted to any of $[-5, -3]$, $[-3, -1]$, or $[-1, 1]$ to obtain an inverse. Each interval has length 2; the choice is not unique without an additional criterion.

8. Self-inverse rational functions

Let $f(x) = \dfrac{3 - 2x}{x + c}$. Solving $y = (3 - 2x)/(x + c)$ for x gives

$$x = \frac{3 - cy}{y + 2}, \quad f^{-1}(x) = \frac{3 - cx}{x + 2}.$$

For a nonconstant self-inverse function, compare with $f(x)$: $\boxed{c = 2}$. The function $f(x) = (3 - 2x)/(x + 2)$ is defined on $\mathbb{R} \setminus \{-2\}$ and is its own inverse.

9. Common pitfalls

- Never write $f^{-1}(x) = 1/f(x)$; this is a reciprocal, not an inverse.
- For x^2 and x^4, choose the correct sign of the root based on the restricted domain.
- State the inverse's domain from the original function's *range*.
- When proving parity, check the domain is symmetric about 0.
- For a self-inverse map, verify both $f(f(x)) = x$ and that the domain is preserved.



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Mathematics made clear.