DP Analysis & Approaches (HL)

Maths with Lemon

Systems of Three Equations – Notes

AHL 1.16	Syllabus focus
Systems of linear equations	A maximum of three equations in three unknowns, including unique solutions, infinitely many solutions and no solution. Algebraic and technological methods, including row reduction and matrices.

1. Systems of Three Linear Equations

A system of three linear equations usually has the form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3. \end{aligned}$$

A solution is an ordered triple

$$\boxed{(x, y, z)}$$

that satisfies all three equations simultaneously.
Geometrically, each equation represents a **plane in three-dimensional space**.

2. Algebraic Elimination

The main idea is to eliminate one variable from two different pairs of equations, producing a system of two equations in two unknowns.

Strategy
1. Label the equations (1), (2), (3).
2. Choose one variable to eliminate.
3. Eliminate it from two pairs of equations.
4. Solve the resulting two-variable system.
5. Substitute back to find the remaining variable.
6. Check the ordered triple in all three original equations.

Worked example

Solve

$$\begin{aligned} -7x - 7y + z &= 5 \quad (1) \\ 4x + 4y - z &= -5 \quad (2) \\ 9x + 10y - z &= -2. \quad (3) \end{aligned}$$

Add (1) and (2):

$$-3x - 3y = 0 \quad \Rightarrow \quad x + y = 0. \tag{4}$$

Subtract (2) from (3):

$$5x + 6y = 3. \tag{5}$$

From (4), $x = -y$. Substitute into (5):

$$5(-y) + 6y = 3 \Rightarrow y = 3.$$

Hence

$$x = -3.$$

Substitute into (2):

$$4(-3) + 4(3) - z = -5 \Rightarrow -z = -5 \Rightarrow z = 5.$$

Therefore

$$\boxed{(x, y, z) = (-3, 3, 5)}.$$

3. Types of Solutions

A. Unique solution

A system has a **unique solution** when elimination gives one specific value for each variable.

$$\boxed{(x, y, z) = (a, b, c)}$$

Graphically, the three planes share exactly one common point.
This is called an

$$\boxed{\text{independent consistent system}}.$$

B. Infinitely many solutions

If elimination produces an identity such as

$$0 = 0,$$

one equation contains no new information. There is at least one free variable.
This is a

$$\boxed{\text{dependent consistent system}}.$$

A general solution must be given using a parameter.
For example, if

$$x = 2 - 3z, \qquad y = 1 + z,$$

let

$$z = t, \qquad t \in \mathbb{R}.$$

Then

$$\boxed{(x, y, z) = (2 - 3t, 1 + t, t), \quad t \in \mathbb{R}}.$$

C. No solution

If elimination produces a contradiction such as

$$0 = 5,$$

the system has no solution.
This is called an

$$\boxed{\text{inconsistent system}}.$$

Graphically, the three planes have no point common to all three.

4. Dependent Systems and General Solutions

Consider

$$\begin{aligned} x + 4y - 2z &= -3, \\ 2x - y + 5z &= 12, \\ 8x + 5y + 11z &= 30. \end{aligned}$$

Elimination shows that one equation is dependent on the others, so one variable is free.
Let

$$z = t.$$

Solving the first two equations gives

$$y = t - 2, \qquad x = 5 - 2t.$$

Hence the general solution is

$$\boxed{(x, y, z) = (5 - 2t, t - 2, t), \quad t \in \mathbb{R}}.$$

5. Inconsistent Systems

Consider

$$\begin{aligned} x + y + z &= 10, \\ x + 2y + 3z &= 4, \\ 2x + 3y + 4z &= 9. \end{aligned}$$

Adding the first two left-hand sides gives the left-hand side of the third:

$$(x + y + z) + (x + 2y + 3z) = 2x + 3y + 4z.$$

But the constants would require

$$10 + 4 = 9,$$

which is false. Thus the system is inconsistent:

$$\boxed{\text{no solution}}.$$

6. Missing Coefficients and Parameters

Parameter questions ask when a system changes from a unique solution to infinitely many solutions or no solution.
The key is to eliminate variables while keeping the parameter.

Example: finding when a system is inconsistent

$$\begin{aligned} x + 2y + kz &= 8, \\ 2x + 5y + 2z &= 7, \\ 5x + 12y + z &= 2. \end{aligned}$$

Eliminating x:

$$\begin{aligned} (2) - 2(1): &\quad y + (2 - 2k)z = -9, \\ (3) - 5(1): &\quad 2y + (1 - 5k)z = -38. \end{aligned}$$

Subtract twice the first reduced equation from the second:

$$(-3 - k)z = -20.$$

If $k \neq -3$, a value of z exists and the system has a unique solution.
If

$$k = -3,$$

the equation becomes

$$0 = -20,$$

a contradiction. Therefore

$$\boxed{k = -3}$$

makes the system inconsistent.

7. Matrix Method

The system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

can be written as

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Equivalently, use the augmented matrix

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right].$$

Elementary row operations are:

$$R_i \leftrightarrow R_j, \qquad R_i \to kR_i, \qquad R_i \to R_i + kR_j.$$

Row reduction can reveal:

$[0\ 0\ 0 \mid d],\ d \neq 0$ $\Rightarrow$ no solution,

$[0\ 0\ 0 \mid 0]$ with a free variable $\Rightarrow$ infinitely many solutions.

8. Applications

Systems of three equations can be used to determine unknown coefficients.
If

$$y = ax^2 + bx + c$$

passes through three known points, substitute each point to obtain three equations in a, b, c.
For example, using $(1, 4), (3, 14), (4, 2.5)$:

$$\begin{aligned} a + b + c &= 4, \\ 9a + 3b + c &= 14, \\ 16a + 4b + c &= 2.5. \end{aligned}$$

Solving this system determines the quadratic.
The same idea applies to models such as

$$y = ax^3 + bx^2 + cx,$$

and to word problems involving three unknown quantities.

9. Summary

Type	Elimination result	Meaning
Unique	specific x, y, z	independent consistent
Infinite	identity such as $0 = 0$	dependent consistent; use a parameter
None	contradiction such as $0 = 5$	inconsistent



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Mathematics made clear.