DP Mathematics: Analysis & Approaches HL

Polynomial Functions – AHL 2.12

Maths with Lemon

Mathematician: _____

Warm Up

Let $f(x) = 2x^2 - 8x + 6$. (a) Find $f(0)$. [1] (b) Solve $f(x) = 0$. [3] (c) Write $f(x) = a(x-h)^2 + k$ and find a, h, k. [5] (d) State the vertex and axis of symmetry. [2] (e) Reflect f in the x-axis, translate right 1 and up 3; find $g(x)$ in standard form. [4]

Polynomial Long Division

Divide and give quotient and remainder. The original lesson groups the left column as "without remainder" and the right as "with remainder". **1.** $(11x^4 + 23x^3 + 24x^2 + 3x - 9) \div (x + 1)$

2. $(11x^5 - 17x^4 - 26x^2 + 3x + 29) \div (x - 2)$

3. $(25x^5 - 16x^4 - 24x^3 + 5x^2 - 24x - 20) \div (6x + 5)$

4. $(8x^2 - 10x - 6) \div (x - 2)$

5. $(x^4 - 3x^3 - 26x^2 + 30x + 12) \div (x^2 - 6x - 2)$

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6. $(x^4 - 12x^2 + 11) \div (x^2 + 2x - 4)$

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Synthetic Division

Worked Example	You Try
$(3x^4 - 24x^3 + 22x^2 - 4x - 21) \div (x - 7)$ Use synthetic division; show the quotient and remainder.	$(2x^2 + 2x - 9) \div (x + 2)$ *Show your working.* ....

Worked Example	You Try
$(x^3 + 4x^2 - 8x + 25) \div (x + 6)$ Use synthetic division; show the quotient and remainder.	$(5x^4 + 20x^3 + 15x^2 - 12) \div (x + 3)$ *Show your working.* ....

Worked Example	You Try
$(12x^2 - 22x + 21) \div (x - 1)$ Use synthetic division; show the quotient and remainder.	$(9x^3 - 30x^2 + 4x + 19) \div (x - 3)$ *Show your working.* ....

Remainder and Factor Theorems

Worked Example	You Try
$f(x) = 3x^3 + 2x - 5, \quad c = 2$ Evaluate $f(c)$. Is $x - c$ a factor?	$f(x) = 3x^4 - x^3 + 2x^2 - 3x + 6, \quad c = -1$ *Show your working.* ....

Worked Example	You Try
$f(x) = 3x^3 + 11x + 2, \quad c = -2$ Evaluate $f(c)$. Is $x - c$ a factor?	$f(x) = x^4 + 3x^3 - 6x^2 + 3x + 2, \quad c = 1$ *Show your working.* ....

Worked Example	You Try
$f(x) = 3x^3 - 7x + 13, \quad c = 1$ Evaluate $f(c)$. Is $x - c$ a factor?	$f(x) = 5x^3 - 5x, \quad c = -2$ *Show your working.* ....

Sum and Product of Quadratic Roots

Worked Example	You Try
Find sum and product of roots: $x^2+12x+14=0$ Apply Vieta's formulas.	Find sum and product of roots: $-3x^2-6x-12=0$ *Show your working.* ....

Worked Example	You Try
Construct a monic quadratic with sum and product: $S = -5,\ P = -20$ Apply Vieta's formulas.	Construct a monic quadratic with sum and product: $S = 4,\ P = 6$ *Show your working.* ....

Worked Example	You Try
Given one root, find the other and k: $k+x^2+x = 0, \quad x_1 = -4$ Apply Vieta's formulas.	Given one root, find the other and k: $k + x^2 - 16x = 0, \quad x_1 = 15$ *Show your working.* ....

Worked Example	You Try
Given one root, find the other and k: $kx + x^2 - 98 = 0, \quad x_1 = 14$ Apply Vieta's formulas.	Given one root, find the other and k: $kx + x^2 - 63 = 0, \quad x_1 = 7$ *Show your working.* ....

Sum and Product of Polynomial Roots

Worked Example	You Try
$2x^3 - 3x^2 + x - 4 = 0$ Find the sum and product of all roots.	$x^3 - 2x^2 - 8x + 11 = 0$ *Show your working.* ....

Worked Example	You Try
$3x^5 - 2x^4 + 3x + 2 = 0$ Find the sum and product of all roots.	$3x^3 - 5x + 3 = 0$ *Show your working.* ....

Worked Example	You Try
$x^3 + 3x - 3 = 0$ Find the sum and product of all roots.	$3x^4 - x^3 + 2x^2 - 3x + 6 = 0$ *Show your working.* ....

Intermediate Problems

Problem 1. When $f(x)$ is divided by $x + 1$, the quotient is $3x^2 + 2x - 1$ and the remainder is 9. Find $f(x)$.

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Problem 2. The remainder of $3x^2 + 7x + k$ on division by $x + 2$ is -10. Find k.

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Problem 3. The remainder of $2x^2 + kx - 3$ on division by $x - 4$ is 9. Find k.

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Problem 4. Given $x - 2$ is a factor of $x^3 - kx^2 - 24x + 28$, find k, all roots, and the full factorization.

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Problem 5. Find the quadratic with roots -1 and $1/3$, whose value at $x = 2$ is 10.

Problem 6. Find the cubic with a root -1, a double root 3, and value 12 at $x = 2$.

Problem 7. For $p(x) = x^3 - 4x^2 + ax - 3$: (a) find a if the remainder on division by $x + 1$ is -12; (b) find all factors.

Problem 8. For $h(x) = 3x^3 - kx^2 - 6x + 8$: (a) given $x - 4$ is a factor, find k and the other factors; (b) find all roots.

Problem 9. Find the quadratic with remainders -6 on division by $x - 1$, -4 by $x - 3$, and 0 by $x + 1$.

Problem 10. Find the cubic with roots $1, 1+\sqrt{2}, 1-\sqrt{2}$, and value -2 at $x=2$.

Problem 11. If α, β solve $3x^2-6x+1=0$, find $\alpha^2+\beta^2$ without solving.

Problem 12. If α, β solve $-2x^2+4x+\frac{5}{2}=0$, find a quadratic whose roots are $\frac{2}{\alpha+1}, \frac{2}{\beta+1}$.

Problem 13. If one root of $x^2-6x+k=0$ is three times the other, find both roots and k.

Problem 14. If $ax+b$ is a factor of x^2+bx+a, express b in terms of a.

Problem 15. Given $2x+1$ and $x-2$ are factors of $2x^4-3x^3+16x^2-27x-18$, find all roots.

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Problem 16. If $f(x) = x^3 + ax^2 + 3x + b$ and the remainder on division by $x + 1$ is 6, find the remainder on division by $x - 1$.

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Detailed Markscheme

Warm Up

(a) $f(0) = 6$. (b) $2(x^2-4x+3) = 2(x-1)(x-3)$, so $x = 1, 3$. (c) $f(x) = 2[(x-2)^2-4]+6 = 2(x-2)^2-2$. Thus $a = 2, h = 2, k = -2$. (d) Vertex $(2,-2)$, axis $x = 2$. (e) Reflection: $-f(x)$; right 1 and up 3: $g(x) = -f(x-1)+3 = -2x^2+12x-11$.

Polynomial Long Division: Quotient and Remainder

1. $Q(x) = 11x^3+12x^2+12x-9, \quad R(x) = 0.$
2. $Q(x) = 11x^4+5x^3+10x^2-6x-9, \quad R(x) = 11.$
3. $Q(x) = \frac{25x^4}{6} - \frac{221x^3}{36} + \frac{241x^2}{216} - \frac{125x}{1296} - \frac{30479}{7776}, \quad R(x) = -\frac{3125}{7776}.$
4. $Q(x) = 8x+6, \quad R(x) = 6.$
5. $Q(x) = x^2+3x-6, \quad R(x) = 0.$
6. $Q(x) = x^2-2x-4, \quad R(x) = -5.$

Synthetic Division: Quotient and Remainder

1. $Q(x) = 3x^3-3x^2+x+3,\ R = 0$. Check: $P = DQ+R$.
2. $Q(x) = 2x-2,\ R = -5$. Check: $P = DQ+R$.
3. $Q(x) = x^2-2x+4,\ R = 1$. Check: $P = DQ+R$.
4. $Q(x) = 5x^3+5x^2,\ R = -12$. Check: $P = DQ+R$.
5. $Q(x) = 12x-10,\ R = 11$. Check: $P = DQ+R$.
6. $Q(x) = 9x^2-3x-5,\ R = 4$. Check: $P = DQ+R$.

Remainder and Factor Theorems

1. $f(2) = 23$. Therefore the divisor is not a factor.
2. $f(-1) = 15$. Therefore the divisor is not a factor.
3. $f(-2) = -44$. Therefore the divisor is not a factor.
4. $f(1) = 3$. Therefore the divisor is not a factor.
5. $f(1) = 9$. Therefore the divisor is not a factor.
6. $f(-2) = -30$. Therefore the divisor is not a factor.

Quadratic Roots

1. $x^2+12x+14 = 0:\ S = -12,\ P = 14. \quad -3x^2-6x-12 = 0:\ S = -2,\ P = 4.$
2. From $x^2-Sx+P = 0$: $x^2+5x-20 = 0$, and $x^2-4x+6 = 0$.
3. $x^2+x+k = 0$: the other root is 3, so $k = (-4)(3) = -12$. For $x^2-16x+k = 0$: other root 1, $k = 15$.
4. $x^2+kx-98 = 0$: other root -7, sum 7, hence $k = -7$. For $x^2+kx-63 = 0$: other root -9, sum -2, hence $k = 2$.

Polynomial Roots: Sum and Product

1a. $S = \frac{3}{2}, \quad P = 2.$
1b. $S = 2, \quad P = -11.$
2a. $S = \frac{2}{3}, \quad P = -\frac{2}{3}.$
2b. $S = 0, \quad P = -1.$
3a. $S = 0, \quad P = 3.$
3b. $S = \frac{1}{3}, \quad P = 2.$

Intermediate Problems: Analytical Solutions

Problem 1. $f(x) = (x+1)(3x^2+2x-1)+9 = \boxed{3x^3+5x^2+x+8}$.

Problem 2. $f(-2) = 12 - 14 + k = k - 2 = -10 \Rightarrow \boxed{k = -8}$.

Problem 3. $f(4) = 32 + 4k - 3 = 9 \Rightarrow 4k = -20 \Rightarrow \boxed{k = -5}$.

Problem 4. $P(2) = 8 - 4k - 48 + 28 = 0 \Rightarrow k = -3$. Thus $P(x) = x^3 + 3x^2 - 24x + 28 = (x-2)(x^2 + 5x - 14) = \boxed{(x-2)^2(x+7)}$. Roots 2 (double), -7.

Problem 5. $P(x) = c(x+1)(x-\frac{1}{3})$. Since $P(2) = c(3)(5/3) = 5c = 10$, $c = 2$. Hence $\boxed{P(x) = 2x^2 + \frac{4}{3}x - \frac{2}{3}}$.

Problem 6. $P(x) = c(x+1)(x-3)^2$. At $x = 2$, $12 = c(3)(1)$, so $c = 4$. Thus $\boxed{4(x+1)(x-3)^2}$.

Problem 7. $p(-1) = -1 - 4 - a - 3 = -8 - a = -12 \Rightarrow a = 4$. Then $p(x) = x^3 - 4x^2 + 4x - 3 = (x-3)(x^2 - x + 1)$. The real linear factor is $x - 3$; the other roots are $(1 \pm i\sqrt{3})/2$.

Problem 8. $h(4) = 192 - 16k - 24 + 8 = 176 - 16k = 0 \Rightarrow k = 11$. Division gives $h(x) = (x-4)(3x^2 + x - 2) = \boxed{(x-4)(3x-2)(x+1)}$. Roots $4, 2/3, -1$.

Problem 9. Let $P(x) = ax^2 + bx + c$. $P(1) = -6$, $P(3) = -4$, $P(-1) = 0$. Thus $a + b + c = -6$, $9a + 3b + c = -4$, $a - b + c = 0$. Solving: $a = 1$, $b = -3$, $c = -4$. So $\boxed{x^2 - 3x - 4}$.

Problem 10. $P(x) = c(x-1)(x-1-\sqrt{2})(x-1+\sqrt{2}) = c(x-1)[(x-1)^2 - 2]$. At $x = 2$, $P(2) = -c = -2 \Rightarrow c = 2$. Thus $\boxed{2(x-1)(x^2 - 2x - 1)}$.

Problem 11. $\alpha + \beta = 2$, $\alpha\beta = 1/3$. Hence $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \boxed{10/3}$.

Problem 12. $\alpha + \beta = 2$, $\alpha\beta = -5/4$. New roots $u = 2/(\alpha+1), v = 2/(\beta+1)$. Their sum $2(\alpha+\beta+2)/(\alpha\beta+\alpha+\beta+1) = 8/(7/4) = 32/7$; product $4/(7/4) = 16/7$. Thus $\boxed{7t^2 - 32t + 16 = 0}$.

Problem 13. Let roots $r, 3r$. Their sum $4r = 6 \Rightarrow r = 3/2$. Other root $9/2$, so $k = (3/2)(9/2) = \boxed{27/4}$.

Problem 14. Assuming $ae0$, the factor $ax + b$ implies root $-b/a$. Substitute: $b^2/a^2 - b^2/a + a = 0$. Multiply by a^2: $b^2(1-a) + a^3 = 0$, so $\boxed{b^2 = a^3/(a-1)}$, or $b = \pm\sqrt{a^3/(a-1)}$ where real. For $a = 0$, the factor is constant b and is not a nonconstant linear factor.

Problem 15. Divide by $(2x+1)(x-2) = 2x^2 - 3x - 2$: $f(x) = (2x+1)(x-2)(x^2+9)$. Thus $\boxed{x = -1/2,\ 2,\ 3i,\ -3i}$.

Problem 16. $f(-1) = -1+a-3+b = a+b-4 = 6 \Rightarrow a+b = 10$. Hence $f(1) = 1+a+3+b = 4+10 = \boxed{14}$.



Maths with Lemon

Mathematics made clear.