DP Mathematics: Analysis & Approaches HL

Polynomial Functions – AHL 2.12

Maths with Lemon

1. Polynomial Functions

A polynomial of degree n is $P(x) = a_n x^n + \cdots + a_1 x + a_0$, with $a_n \neq 0$. The **zeros** solve $P(x) = 0$; the corresponding x-intercepts are real zeros. If r is a root, then $x - r$ is a factor. A repeated root has multiplicity greater than one. For even multiplicity, the graph touches the x-axis; for odd multiplicity, it crosses it.

2. Polynomial Long Division

Use the division identity

$$\boxed{P(x) = D(x)Q(x) + R(x)}, \qquad \deg R < \deg D.$$

Arrange terms in descending powers, insert missing zero coefficients, divide leading terms, multiply, subtract and repeat.

Worked example:

$$\frac{x^3 - 2x^2 - 5x + 6}{x - 3}.$$

First $x^3/x = x^2$. Subtract $x^3 - 3x^2$, leaving $x^2 - 5x + 6$. Next $x^2/x = x$. Subtract $x^2 - 3x$, leaving $-2x + 6$. Finally $-2x/x = -2$, leaving remainder 0.

$$\boxed{x^3 - 2x^2 - 5x + 6 = (x - 3)(x^2 + x - 2)}.$$

A nonzero remainder: $x^2 + 1 = (x - 2)(x + 2) + 5$.

3. Synthetic Division

For division by $x - c$, write coefficients (including zeros), bring down the first, then multiply by c and add repeatedly.

3	1	−2	−5	6
		3	3	−6
	1	1	−2	0

Thus the quotient is $x^2 + x - 2$ and the remainder is zero. For $x + 2$, use $c = -2$, *not* 2.

4. Remainder and Factor Theorems

When $P(x)$ is divided by $x - c$, the remainder is $P(c)$. Consequently, $x - c$ is a factor exactly when $P(c) = 0$.

$$P(x) = 2x^3 - 3x^2 - 3x + 2, \qquad P(2) = 16 - 12 - 6 + 2 = 0.$$

Therefore $x - 2$ is a factor. Synthetic division gives

$$P(x) = (x - 2)(2x^2 + x - 1) = (x - 2)(2x - 1)(x + 1).$$

Tip: To find an unknown coefficient, substitute the known root or remainder.

5. Sum and Product of Roots

For $ax^2 + bx + c = 0$, roots α, β:

$$\boxed{\alpha + \beta = -\frac{b}{a}}, \qquad \boxed{\alpha\beta = \frac{c}{a}}.$$

For $ax^3 + bx^2 + cx + d = 0$, roots α, β, γ:

$$\alpha + \beta + \gamma = -\frac{b}{a}, \qquad \alpha\beta\gamma = -\frac{d}{a}.$$

For degree n, counting complex roots and multiplicities:

$$\boxed{\sum_{i=1}^{n} r_i = -\frac{a_{n-1}}{a_n}}, \qquad \boxed{\prod_{i=1}^{n} r_i = (-1)^n \frac{a_0}{a_n}}.$$

These are consequences of Vieta's formulas.
Worked example: For $2x^3 - 3x^2 + x - 4 = 0$, the sum of the three roots is $3/2$ and their product is 2. For $x^2 + 5x - 14 = 0$, the roots have sum -5 and product -14.

6. Constructing Polynomials from Roots

If the roots are $r_1, \ldots, r_n$, then

$$P(x) = k(x - r_1) \cdots (x - r_n).$$

Use a known value $P(t)$ to determine k. For example, a quadratic with roots -1 and 3, and $P(0) = 6$, is

$$P(x) = k(x+1)(x-3), \quad 6 = -3k, \quad \boxed{P(x) = -2(x+1)(x-3)}.$$

7. Transformations and Symmetric Expressions

If $\alpha + \beta = S$ and $\alpha\beta = P$, then

$$\alpha^2 + \beta^2 = S^2 - 2P, \qquad (\alpha + 1)(\beta + 1) = P + S + 1.$$

To build a quadratic with new roots u, v, use

$$t^2 - (u + v)t + uv = 0.$$

Worked example: If $3x^2 - 6x + 1 = 0$, then $S = 2,\ P = 1/3$, so

$$\boxed{\alpha^2 + \beta^2 = 4 - \frac{2}{3} = \frac{10}{3}}.$$

8. Exam Checklist

1. Write the polynomial in descending powers, including missing zero terms.
2. Check divisor sign carefully in synthetic division.
3. Use $P(c)$ for the remainder; $P(c) = 0$ for a factor.
4. Check whether a problem needs *all* roots, including complex roots.
5. In root-sum problems, remember multiplicities and the leading coefficient.
6. Verify a factorization by expanding.



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