Mclaurin [233 marks]

1. [Maximum mark: 19] 26M.1.AHL.TZ3.12

Consider $y = \tan\left(1 - \mathrm{e}^{3x}\right)$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -3\mathrm{e}^{3x}\left(1 + y^2\right)$. [4]

(b) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9$, when $x = 0$. [5]

The function f is defined by $f(x) = \tan\left(1 - \mathrm{e}^{3x}\right)$, where $-0.2 \leq x \leq 0.2$.

(c) Find the Maclaurin series for $f(x)$, up to and including the term in x^2. [2]

(d) Hence, or otherwise, find $\lim\limits_{x \to 0} \frac{\tan\left(1-\mathrm{e}^{3x}\right)}{x}$. [3]

The answer to part (c) is equal to the first two non-zero terms in the series expansion of $\frac{x}{a+bx}$, where $a,\ b \in \mathbb{Q}$.

(e) Find the value of a and the value of b. [5]

2. [Maximum mark: 7] 25N.1.AHL.TZ1.5

The first four terms of the Maclaurin series expansion of $(1 - x)^{-4}$ are

$1 + ax + bx^2 + 20x^3, a, b \in \mathbb{Z}^+$.

(a.i) Show that $a = 4$. [2]

(a.ii) Find the value of b. [3]

A car was purchased four years ago. The car depreciated in value by 10% each year. The value of the car today is $\$1000$.

(b) Using the results of part (a), estimate the value of the car four years ago. [2]

3. [Maximum mark: 18] 25N.1.AHL.TZ1.10

Consider the expression $\frac{1}{1+x^2}$, where $|x| < 1$.

(a.i) Show that the first four terms in the binomial expansion of $\frac{1}{1+x^2}$ are $1 - x^2 + x^4 - x^6$. [3]

(a.ii) Hence, find an approximation for $\int \frac{1}{1+x^2}\mathrm{d}x$ up to and including the term in x^7. [2]

Now consider the expression $\frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$.

(b.i) Expand $\frac{1}{\sqrt{1-x^2}}$ up to and including the term in x^4. [4]

(b.ii) Hence, find a polynomial expression for $\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$ up to and including the term in x^5. [1]

The expression $\frac{25}{48} + \frac{k}{1280}$, where $k \in \mathbb{Z}^+$, can be used to approximate $\arcsin \frac{1}{2}$.

(c) Use the result from part (b)(ii) to find the value of k. [4]

(d) Hence, find an approximation for π in the form $\frac{p}{640}$ where $p \in \mathbb{Z}^+$. [4]

4. [Maximum mark: 19] 25M.1.AHL.TZ1.12

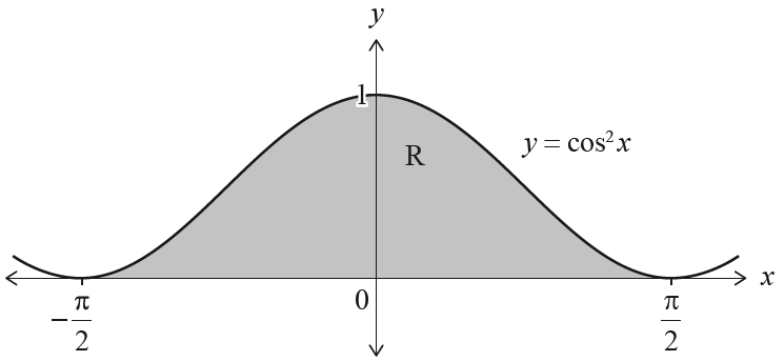
Consider the family of functions $f_n(x) = \cos^n x$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) By writing $\cos^n x$ as $\cos^{n-1} x \cos x$, show that $\int \cos^n x \ \mathrm{d}\ x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \ \mathrm{d}\ x - (n-1)\int \cos^n x \ \mathrm{d}\ x$ for $n > 1$. [4]

(b) Hence, show that $\int f_n(x) \ \mathrm{d}\ x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int f_{n-2}(x) \ \mathrm{d}\ x$ for $n > 1$. [2]

(c) Hence, find an expression for $\int \cos^4 x \ \mathrm{d}\ x$, giving your answer in the form $p \cos^3 x \sin x + q \cos x \sin x + rx + c$ where $p,\ q,\ r \in \mathbb{Q}^+$. [4]

The region R is enclosed by the graph of $y = \cos^2 x$ and the x-axis where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, as shown in the following diagram.



The region R is rotated by 2π radians around the x-axis to form a solid of revolution.

(d) Find the volume of the solid. [4]

(e.i) Find the Maclaurin series of $f_n(x)$ up to the term in x^2. [3]

(e.ii) Hence or otherwise, find $\lim_{x \to 0} \frac{f_n(x)-1}{x^2}$ in terms of n. [2]

5. [Maximum mark: 17] 24N.1.AHL.TZ0.11

A curve is given by the equation $y = \frac{e^{2x}-1}{e^{2x}+1}$, $x \in \mathbb{R}$.

(a) By applying l'Hôpital's rule or otherwise, show that $\lim_{x \to \infty} \left(\frac{e^{2x}-1}{e^{2x}+1} \right) = 1$. [2]

(b.i) Show that $\frac{dy}{dx} = \frac{4e^{2x}}{(e^{2x}+1)^2}$. [3]

(b.ii) Hence, show that $1 - y^2 = \frac{dy}{dx}$. [3]

(c.i) By using implicit differentiation and the result in part (b)(ii), show that $\frac{d^2y}{dx^2} = 2y^3 - 2y$. [3]

(c.ii) Hence, find an expression for $\frac{d^3y}{dx^3}$ in terms of y. [2]

(d) By using your results from parts (b) and (c), find the Maclaurin series for $\frac{e^{2x}-1}{e^{2x}+1}$ up to and including the term in x^3. [4]

6. [Maximum mark: 7] 24M.1.AHL.TZ1.8

(a) Find the first two non-zero terms in the Maclaurin series of

(a.i) $\sin\left(x^2\right)$; [2]

(a.ii) $\sin^2\left(x^2\right)$. [3]

(b) Hence, or otherwise, find the first two non-zero terms in the Maclaurin series of $4x\sin\left(x^2\right)\cos\left(x^2\right)$. [2]

7. [Maximum mark: 20] 24M.1.AHL.TZ2.12

(a) Let $f(x) = (1-ax)^{-\frac{1}{2}}$, where $ax < 1,\ a \neq 0$.

The n^{th} derivative of $f(x)$ is denoted by $f^{(n)}(x),\ n \in \mathbb{Z}^+$.

Prove by induction that $f^{(n)}(x) = \frac{a^n(2n-1)!(1-ax)^{-\frac{2n+1}{2}}}{2^{2n-1}(n-1)!},\ n \in \mathbb{Z}^+$. [8]

(b) By using part (a) or otherwise, show that the Maclaurin series for $f(x) = (1-ax)^{-\frac{1}{2}}$ up to and including the x^2 term is $1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2$. [2]

(c) Hence, show that $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \frac{2+6x+19x^2}{2}$. [4]

(d) Given that the series expansion for $(1-ax)^{-\frac{1}{2}}$ is convergent for $|ax| < 1$, state the restriction which must be placed on x for the approximation $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \frac{2+6x+19x^2}{2}$ to be valid. [1]

(e) Use $x = \frac{1}{10}$ to determine an approximate value for $\sqrt{3}$.

Give your answer in the form $\frac{c}{d}$, where $c,\ d \in \mathbb{Z}^+$. [5]

8. [Maximum mark: 21] 23N.1.AHL.TZ1.11

Consider the function $f(x) = \mathrm{e}^{\cos 2x}$, where $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

(c) Sketch the curve of $y = f(x)$ for $0 \leq x \leq \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

(d.i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4. [2]

(d.ii) Hence, find the Maclaurin series for $\mathrm{e}^{\cos^{2x-1}}$, up to and including the term in x^4. [3]

(d.iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [1]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} \mathrm{e}^{\cos 2x} \mathrm{d}x \approx \frac{149\mathrm{e}}{1500}$ [3]

9. [Maximum mark: 21] 23N.1.AHL.TZ2.11

Consider the function $f(x) = \mathrm{e}^{\cos 2x}$, where $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

(c) Sketch the curve of $y = f(x)$ for $0 \leq x \leq \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

(d.i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4. [2]

(d.ii) Hence, find the Maclaurin series for $\mathrm{e}^{\cos^2 x - 1}$, up to and including the term in x^4. [3]

(d.iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [1]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} \mathrm{e}^{\cos 2x} \mathrm{d}x \approx \frac{149\mathrm{e}}{1500}$ [3]

10. [Maximum mark: 21] 22M.1.AHL.TZ1.12

The function f is defined by $f(x) = \mathrm{e}^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the x^3 term. [4]

(b) Hence, find an approximate value for $\int_0^1 \mathrm{e}^{x^2} \sin\left(x^2\right) \mathrm{d}\, x$. [4]

The function g is defined by $g(x) = \mathrm{e}^x \cos x$, where $x \in \mathbb{R}$.

(c.i) Show that $g(x)$ satisfies the equation $g''(x) = 2(g'(x) - g(x))$. [4]

(c.ii) Hence, deduce that $g^{(4)}(x) = 2(g'''(x) - g''(x))$. [1]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the x^4 term. [5]

(e) Hence, or otherwise, determine the value of $\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3}$. [3]

11. [Maximum mark: 14] 21N.1.AHL.TZ0.11

(a) Prove by mathematical induction that $\frac{d^n}{dx^n}\left(x^2 e^x\right) = \left[x^2 + 2nx + n(n-1)\right]e^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x) = x^2 e^x$ in ascending powers of x, up to and including the term in x^4. [3]

(c) Hence or otherwise, determine the value of $\lim_{x \to 0}\left[\frac{\left(x^2 e^x - x^2\right)^3}{x^9}\right]$. [4]

12. [Maximum mark: 20] 21M.1.AHL.TZ1.12

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$ for $n \in \mathbb{Z},\ n \geq 2$. [9]

(c) Let $g(x) = e^{mx},\ m \in \mathbb{Q}$.

Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x > -1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\frac{7}{4}$.

Find the possible values of m. [8]

13. [Maximum mark: 8] 21M.2.AHL.TZ2.9

(a) Write down the first three terms of the binomial expansion of $(1+t)^{-1}$ in ascending powers of t. [1]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1 + \frac{x^2}{2} + \frac{5x^4}{24}$. [4]

(c) By using the Maclaurin series for $\arctan x$ and the result from part (b), find $\lim_{x \to 0}\left(\frac{x \arctan 2x}{\sec x - 1}\right)$. [3]

14. [Maximum mark: 21] SPM.1.AHL.TZ0.12

The function f is defined by $f(x) = \mathrm{e}^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x} - 1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x} - 1\right)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim\limits_{x \to 0} \frac{f(x) - 1}{\arctan(\mathrm{e}^{3x} - 1)}$. [3]

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