DP Analysis & Approaches HL

Maths with Lemon

Lesson 9.3HL: Volumes of Revolution — Theory Notes

1. From Area to Volume

Before studying volumes of revolution, recall that integration can be used to calculate the area under a curve.
If

$$y = f(x)$$

and the region lies between $x = a$ and $x = b$, then

$$\boxed{A = \int_a^b f(x)\,dx}$$

provided that $f(x) \geq 0$ on the interval.
For a region between two curves,

$$y = f(x)$$

and

$$y = g(x),$$

where $f(x) \geq g(x)$,

$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx.}$$

Volumes of revolution extend this idea from a **two-dimensional area** to a **three-dimensional solid**.

2. What is a Solid of Revolution?

When a region is rotated through

$$360^\circ$$

or

$$2\pi \text{ radians}$$

around a line, it produces a three-dimensional solid.
This is called a

$$\boxed{\text{solid of revolution}}.$$

The volume of the solid is called a

$$\boxed{\text{volume of revolution}}.$$

For example, imagine rotating the region under

$$y = f(x)$$

around the x-axis.
Each vertical strip produces a thin circular disk.

3. Why Does the Formula Contain πy^2?

Consider a very thin vertical strip of width

$$dx.$$

If the height of the curve is y, rotating this strip around the x-axis creates a thin disk. The radius of the disk is

$$r = y.$$

The area of the circular cross-section is therefore

$$A = \pi r^2.$$

Since

$$r = y,$$

we obtain

$$A = \pi y^2.$$

The small volume is approximately

$$dV = \pi y^2 \, dx.$$

Adding infinitely many thin disks gives

$$\boxed{V = \int_a^b \pi y^2 \, dx.}$$

If

$$y = f(x),$$

then

$$\boxed{V = \pi \int_a^b [f(x)]^2 \, dx.}$$

Key Idea

When rotating around the x-axis:

$$\boxed{V = \pi \int_a^b y^2 \, dx}$$

or

$$\boxed{V = \pi \int_a^b [f(x)]^2 \, dx.}$$

4. Rotation About the x-axis

Suppose

$$y = x^3, \qquad 0 \le x \le 2.$$

The radius of each disk is

$$r = x^3.$$

Therefore,

$$V = \pi \int_0^2 (x^3)^2 \, dx.$$

So

$$V = \pi \int_0^2 x^6 \, dx.$$

Integrating,

$$V = \pi \left[\frac{x^7}{7} \right]_0^2.$$

Thus

$$V = \pi \left(\frac{128}{7} \right).$$

Therefore,

$$\boxed{V = \frac{128\pi}{7}}.$$

Notice the important step:

$$(x^3)^2 = x^6.$$

5. A Common Mistake

For rotation around the x-axis, do **not** write

$$V = \pi \int_a^b f(x) \, dx.$$

The function must be squared.
The correct formula is

$$\boxed{V = \pi \int_a^b [f(x)]^2 \, dx.}$$

This happens because we are integrating the areas of circles:

$$A = \pi r^2.$$

6. Rotation About the y-axis

If we use horizontal slices and rotate them around the y-axis, the radius is measured horizontally.
Therefore,

$$r = x.$$

The cross-sectional area is

$$A = \pi x^2.$$

Hence,

$$\boxed{V = \pi \int_c^d x^2 \, dy.}$$

This means that x should normally be written as a function of y.

Key Idea

Around the y-axis:

$$\boxed{V = \pi \int_c^d x^2 \, dy.}$$

The integral is with respect to y.

7. Rewriting x in Terms of y

Suppose

$$y = x^2 - 1, \qquad x \geq 0.$$

To rotate around the y-axis, rearrange:

$$y + 1 = x^2.$$

Therefore,

$$x = \sqrt{y+1}.$$

But since the formula contains x^2, we can use directly

$$x^2 = y + 1.$$

If

$$1 \leq x \leq 5,$$

then the corresponding y-values are

$$y(1) = 0$$

and

$$y(5) = 24.$$

Therefore,

$$V = \pi \int_0^{24} (y+1) \, dy.$$

Integrating,

$$V = \pi \left[\frac{y^2}{2} + y \right]_0^{24}.$$

Hence,

$$V = \pi(288 + 24).$$

Therefore,

$$\boxed{V = 312\pi.}$$

8. Which Variable Should I Integrate With?

A useful rule is:

$$\boxed{\begin{array}{lcl} \text{Rotation about } x\text{-axis} & \Rightarrow & dx \\ \text{Rotation about } y\text{-axis} & \Rightarrow & dy \end{array}}$$

for the disk/washer method used in this lesson.

Rotation	**Formula**
Around x-axis	$V = \pi \int y^2 \, dx$
Around y-axis	$V = \pi \int x^2 \, dy$

9. Determining the Limits

Always check what the limits represent.
For an integral

$$\int_a^b \cdots \, dx,$$

the limits must be x-values.
For an integral

$$\int_c^d \cdots \, dy,$$

the limits must be y-values.
For example, if

$$y = x^2 - 1$$

and

$$1 \le x \le 5,$$

then for integration with respect to y we convert the limits:

$$x = 1 \Rightarrow y = 0,$$

$$x = 5 \Rightarrow y = 24.$$

10. Solids Created from Multiple Curves

Sometimes a region is bounded by more than one curve.
When the region is rotated, the radius of the solid may change from one function to another.
For example, suppose a region is bounded by

$$y = \sin x,$$

$$y = \cos x,$$

and the x-axis.
First determine where the curves intersect:

$$\sin x = \cos x.$$

Therefore,

$$\tan x = 1.$$

Hence,

$$x = \frac{\pi}{4}$$

in the first quadrant.
If different curves form the upper boundary on different intervals, the volume must be split into separate integrals.
For example,

$$V = \pi \int_a^{\pi/4} (\sin x)^2 \, dx + \pi \int_{\pi/4}^{b} (\cos x)^2 \, dx.$$

Key Idea

If the boundary changes, split the integral at the point where the curves intersect.

11. Regions Between Two Curves: Washers

If a region between two curves is rotated, a hole may appear inside the solid.
The cross-section is then a **washer** rather than a disk.
If

$$R = \text{outer radius}$$

and

$$r = \text{inner radius},$$

then the area of the washer is

$$A = \pi R^2 - \pi r^2.$$

Therefore,

$$\boxed{A = \pi(R^2 - r^2).}$$

The volume is

$$\boxed{V = \pi \int_a^b \left(R^2 - r^2\right)\, dx.}$$

12. Determining the Bounded Region

Before setting up a volume integral, determine where the curves intersect. Consider

$$y = x^2$$

and

$$y = \sqrt{x}.$$

At the intersections,

$$x^2 = \sqrt{x}.$$

Squaring gives

$$x^4 = x.$$

Thus,

$$x(x^3 - 1) = 0.$$

Therefore,

$$x = 0$$

or

$$x = 1.$$

Hence the finite region is bounded by

$$0 \leq x \leq 1.$$

On this interval,

$$\sqrt{x} \geq x^2.$$

If the region is rotated around the x-axis,

$$R = \sqrt{x}$$

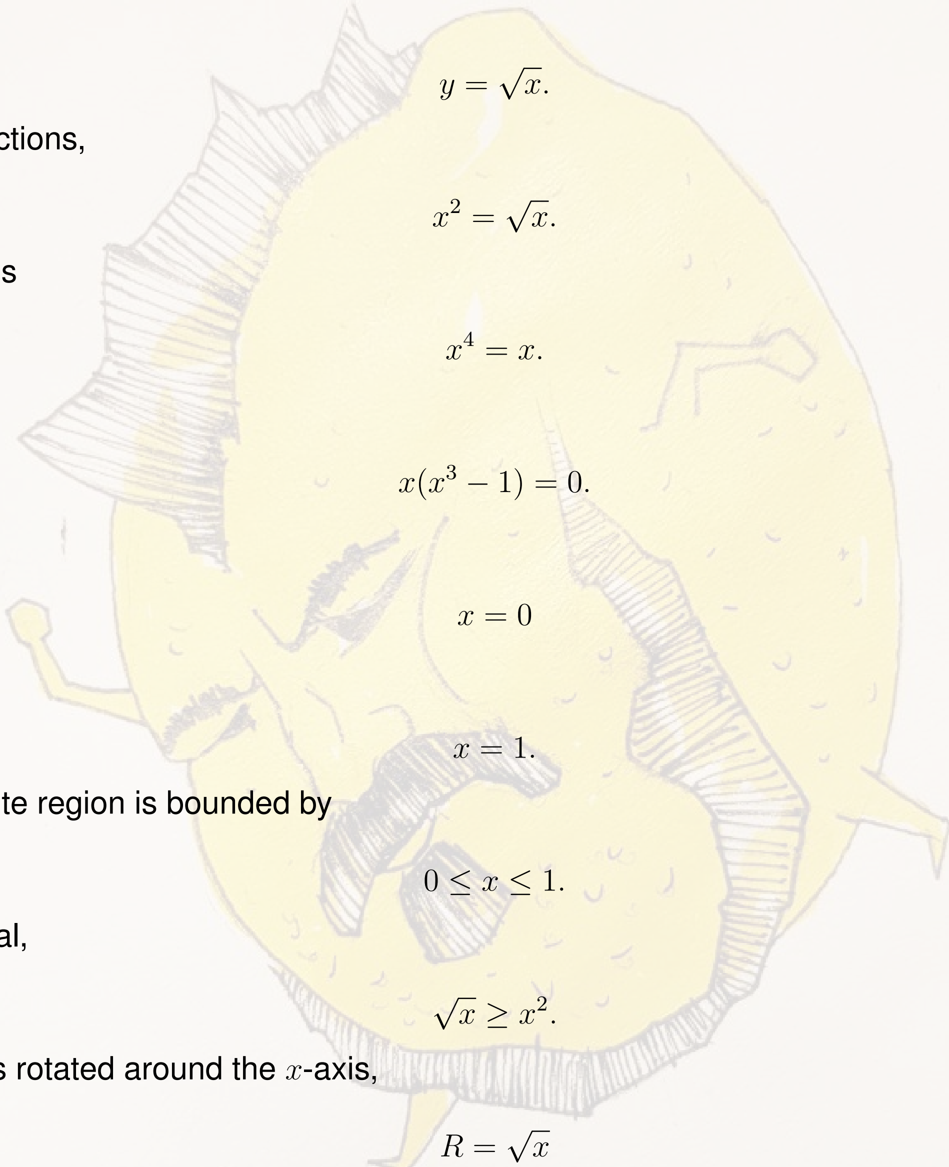
and

$$r = x^2.$$

Therefore,

$$V = \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] dx.$$

Thus,



$$V = \pi \int_0^1 (x - x^4)\, dx.$$

Integrating,

$$V = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1.$$

Hence,

$$V = \pi \left(\frac{1}{2} - \frac{1}{5} \right)$$

and therefore

$$\boxed{V = \frac{3\pi}{10}.}$$

13. Rotation About a Vertical Line

The axis of rotation does not have to be an axis.
For example, a region might be rotated around

$$x = -2.$$

The radius is now the **horizontal distance** from the curve to the line $x = -2$.
If a point has coordinate x, then

$$r = x - (-2).$$

Therefore,

$$\boxed{r = x + 2}.$$

If using washers with respect to y, identify the outer and inner horizontal distances from the axis of rotation.

> **Key Idea**
>
> Radius means **distance from the axis of rotation**.
> It is not automatically x or y.

14. Rotation About a Horizontal Line

Suppose a region is rotated around

$$y = -1.$$

If the curve is

$$y = f(x),$$

then the distance from the curve to the axis is

$$f(x) - (-1).$$

Therefore,

$$\boxed{R = f(x) + 1.}$$

If the region extends from the axis itself to the curve, then

$$V = \pi \int_a^b [f(x) + 1]^2 \, dx.$$

If there is an inner boundary, use the washer formula

$$\boxed{V = \pi \int_a^b (R^2 - r^2) \, dx.}$$

15. Example: Rotation Around $y = -1$

Suppose

$$y = \cos x$$

is rotated around

$$y = -1.$$

The radius from the curve to the axis is

$$R = \cos x + 1.$$

Therefore, for limits a and b,

$$V = \pi \int_a^b (\cos x + 1)^2 \, dx.$$

Expand:

$$(\cos x + 1)^2 = \cos^2 x + 2\cos x + 1.$$

Using

$$\cos^2 x = \frac{1 + \cos 2x}{2},$$

the integral can then be evaluated exactly.

16. Exact Answers

Volume-of-revolution questions frequently ask for an answer **in terms of** π. For example,

$$\boxed{V = \frac{128\pi}{7}}$$

should normally be left in this exact form rather than converted to a decimal. Similarly, expressions involving

$$e, \qquad \ln x, \qquad \pi$$

should remain exact unless a decimal approximation is requested.

17. Common Mistakes

- Forgetting to square the radius.
  Incorrect:

$$V = \pi \int f(x)\,dx.$$

  Correct:

$$\boxed{V = \pi \int [f(x)]^2\,dx.}$$

- Forgetting the factor of π.
- Using dx when the formula has been set up in terms of y.
- Using x-limits in an integral with respect to y.
- Forgetting to find the intersection points of two curves.
- Using

$$R - r$$

  instead of

$$R^2 - r^2.$$

- Forgetting that the radius is the **distance** from the axis of rotation.
- When rotating around $y = -1$, using $f(x)$ instead of

$$f(x) + 1.$$

- When rotating around $x = -2$, using x instead of the distance

$$x + 2.$$

18. Formula Summary

Situation	Formula
Area under a curve	$A = \int_a^b f(x)\,dx$
About the x-axis	$\boxed{V = \pi \int_a^b y^2\,dx}$
About the y-axis	$\boxed{V = \pi \int_c^d x^2\,dy}$
Washer method	$\boxed{V = \pi \int (R^2 - r^2)\,dx}$ or the corresponding integral with respect to y.
Vertical non-axis line	Radius = horizontal distance from the axis.
Horizontal non-axis line	Radius = vertical distance from the axis.

19. Final Checklist

Before evaluating a volume of revolution:

1. What is the axis of rotation?
2. Should I integrate with respect to x or y?
3. What is the radius?
4. Is there an inner radius?
5. Do I need a disk or a washer?
6. Are my limits in the correct variable?
7. Does the boundary change during the interval?
8. If so, do I need to split the integral?
9. Have I squared both radii?
10. Have I included π?
11. Does the final answer have cubic units?

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Mathematics made clear.