Lesson 10.4 – Integrating Factor

DP Mathematics: Analysis and Approaches HL
AHL 5.18 – Theory and Worked Examples

Syllabus	Guidance
AHL 5.18	Solution of $y' + P(x)y = Q(x)$ using an integrating factor.

1. Recognising a first-order linear equation

The standard form is

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x).}$$

The coefficient of y' must be 1. Divide by its coefficient first, if necessary. The functions P, Q depend on x, not y. Work on an interval where they are continuous.

2. Why the integrating factor works

The product rule says

$$\frac{d}{dx}(\mu y) = \mu y' + \mu' y.$$

Multiplying the equation by μ gives $\mu y' + \mu P y = \mu Q$. To match the product rule we need

$$\mu' = P\mu, \qquad \frac{\mu'}{\mu} = P.$$

Integrating, $\ln|\mu| = \int P(x)\,dx$, so a suitable choice is

$$\boxed{\mu(x) = e^{\int P(x)\,dx}.}$$

Consequently

$$\boxed{\frac{d}{dx}(\mu y) = \mu Q, \qquad y = \frac{\int \mu Q\,dx + C}{\mu}.}$$

A nonzero constant multiple of μ gives the same solutions.

3. A reliable six-step method

1. Rewrite in standard form $y' + Py = Q$.
2. Identify $P(x)$ and $Q(x)$.
3. Calculate $\mu = e^{\int P\,dx}$.
4. Multiply the entire equation by μ.
5. Recognise $(\mu y)' = \mu Q$ and integrate.
6. Divide by μ, determine C if an initial condition is given, and check the domain.

4. Worked example from the lesson

Solve $y' + \frac{2}{x}y = x^3$ for $x > 0$.

$$P(x) = \frac{2}{x}, \qquad \mu = e^{\int 2/x\,dx} = e^{2\ln x} = x^2.$$

$$x^2y' + 2xy = x^5 \quad \Longrightarrow \quad (x^2y)' = x^5.$$

$$x^2y = \frac{x^6}{6} + C \quad \Longrightarrow \quad \boxed{y = \frac{x^4}{6} + \frac{C}{x^2}}.$$

5. Exponential forcing and resonance

For $y' + y = e^x$, $\mu = e^x$ and

$$(e^x y)' = e^{2x} \Longrightarrow \boxed{y = \tfrac{1}{2}e^x + Ce^{-x}}.$$

For $y' - y = e^x$, $\mu = e^{-x}$ and

$$(e^{-x}y)' = 1 \Longrightarrow \boxed{y = e^x(x + C)}.$$

The factor x in the second answer is important: the forcing term is already a solution of the corresponding homogeneous equation.

6. Power functions and logarithmic integrating factors

If $P = a/x$, then $\mu = e^{a\ln|x|} = |x|^a$ on an interval not containing 0. In common integer-power examples we can take $\mu = x^a$. For $y' - y/x = x$, use $\mu = 1/x$ on $x > 0$:

$$\left(\frac{y}{x}\right)' = 1 \Longrightarrow \frac{y}{x} = x + C \Longrightarrow \boxed{y = x^2 + Cx}.$$

7. Trigonometric integrating factors

On an interval where $\cos x > 0$,

$$\mu = e^{\int \tan x\,dx} = e^{-\ln(\cos x)} = \sec x.$$

For $y' + y\tan x = 2\sin x$,

$$(y\sec x)' = 2\tan x \Longrightarrow y\sec x = -2\ln|\cos x| + C,$$

$$\boxed{y = \cos x\big(C - 2\ln|\cos x|\big)}.$$

8. Initial conditions

For $y' + \frac{2}{x}y = 12x^2$, $x > 0$, we obtain

$$(x^2y)' = 12x^4, \quad y = \frac{12}{5}x^3 + \frac{C}{x^2}.$$

If $y(1) = 5$, then $C = 13/5$, hence

$$\boxed{y = \frac{12}{5}x^3 + \frac{13}{5x^2}}.$$

9. Substitutions that create linear equations

Some equations are not linear in y but become linear after a substitution.

- If $3xy^2y' + y^3 = e^x$, let $z = y^3$ so $z' = 3y^2y'$, giving $xz' + z = e^x$.
- If $2yy' - y^2/x = x^2$, let $z = y^2$ so $z' = 2yy'$, giving $z' - z/x = x^2$.
- If $y' = (x^2 + y^2)/(2xy)$ and $y > 0$, let $u = y^2$ so $u' = 2yy'$, giving $u' - u/x = x$.
- If $\cos y\,y' + \tan x\sin y = 2\cos^2 x$, let $z = \sin y$, giving $z' + z\tan x = 2\cos^2 x$.

10. A second-order equation reduced to first order

For $y'' - y'/x = x$, let $z = xy'$. Then $z' = y' + xy''$, and

$$z' - \frac{2}{x}z = x^2.$$

Solve this first-order linear equation for z, then recover y' and integrate again. A second-order equation generally requires two arbitrary constants.

11. Modelling: velocity and terminal behaviour

For $v' = 10 - 2v$ with $v(0) = v_0$, use $\mu = e^{2t}$:

$$\boxed{v(t) = 5 + (v_0 - 5)e^{-2t}}.$$

The terminal velocity is $5\,\mathrm{m\,s^{-1}}$ because $e^{-2t} \to 0$. For $v' + v/(t+1) = 10$, use $\mu = t + 1$:

$$\boxed{v(t) = 5(t+1) + \frac{v_0 - 5}{t+1}}.$$

Therefore $v'(t) = 5 - (v_0 - 5)/(t+1)^2 \to 5\,\mathrm{m\,s^{-2}}$. This second model has a limiting acceleration, not a finite terminal speed.

12. Frequent mistakes

- Not dividing through before identifying $P(x)$.
- Multiplying only one term by μ.
- Forgetting the product rule and the integration constant.
- Ignoring restrictions such as $x \neq 0$ or $\cos x \neq 0$.
- Losing the sign when selecting a square-root branch from an initial condition.



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