Lesson 10.3: Separable and Homogeneous Differential Equations

DP Mathematics Analysis & Approaches HL – Theory Notes

AHL 5.18: Separation of variables, the logistic equation, and homogeneous differential equations of the form $y' = f(y/x)$ using $y = vx$. Links: partial fractions (AHL 1.11 and 5.15).

1. Recognising separable equations

If $\frac{dy}{dx} = f(x)g(y)$, move all terms involving y to the left and all terms involving x to the right:

$$\boxed{\int \frac{1}{g(y)}\,dy = \int f(x)\,dx.}$$

Integrate both sides and include one constant C. When dividing by $g(y)$, check whether $g(y) = 0$ yields additional constant solutions. **2. Worked example: $y' = xy$, $y > 0$**

$$\frac{dy}{y} = x\,dx \implies \ln y = \frac{x^2}{2} + C \implies \boxed{y = Ae^{x^2/2},\ A > 0}.$$

3. Other useful separations

$$y' = x^3y^3 \implies y^{-3}dy = x^3dx \implies -\frac{1}{2y^2} = \frac{x^4}{4} + C.$$

For nonzero solutions, $y^{-2} = C - \frac{x^4}{2}$. Also $y = 0$ is a solution lost on dividing by y^3.

$$y' = \frac{\sin x}{y} \implies y\,dy = \sin x\,dx \implies \boxed{y^2 = C - 2\cos x}.$$

4. Initial conditions and domain restrictions

For $y' = 2y$ with $y(0) = 3$, separation gives $\ln|y| = 2x + C$, so $y = Ae^{2x}$ and $\boxed{y = 3e^{2x}}$. When taking square roots, choose the sign consistent with the initial condition and the interval of validity.

5. Homogeneous differential equations

A first-order equation is homogeneous in this sense when $\frac{dy}{dx} = F(y/x)$. Put

$$\boxed{y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}}.$$

Then solve the resulting separable equation for v, before substituting $v = y/x$ back. Work on intervals with $x \neq 0$. **6. Worked homogeneous example**

$$\frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x}, \qquad x > 0.$$

Put $y = vx$. Then $v + xv' = 1 + v$, so $xv' = 1$ and

$$v = \ln x + C \implies \boxed{y = x(\ln x + C)}.$$

7. Worked example with a reciprocal

$$y' = \frac{y}{x} + \frac{x}{y}, \quad y = vx \implies v + xv' = v + \frac{1}{v}.$$

Thus $v\,dv = dx/x$, giving $v^2 = 2\ln|x| + C$ and

$$\boxed{y^2 = x^2(2\ln|x| + C)}.$$

8. Logistic growth and partial fractions

For $\frac{dN}{dt} = kN(a - N)$ with $a, k > 0$, separate:

$$\int \frac{dN}{N(a-N)} = \int k\,dt, \qquad \frac{1}{N(a-N)} = \frac{1}{a}\left(\frac{1}{N} + \frac{1}{a-N}\right).$$

Hence

$$\frac{1}{a}\ln\left|\frac{N}{a-N}\right| = kt + C, \qquad \boxed{N(t) = \frac{a}{1 + Be^{-akt}}}.$$

Here a is the carrying capacity and $B = (a - N_0)/N_0$ when $0 < N_0 < a$. The equilibrium solutions $N = 0$ and $N = a$ must also be considered. **9. Modelling applications**

Radioactive decay: $m' = -km$. If $m(0) = 25$ g and $m'(0) = -5$ g/s, then $k = 1/5$ and $m = 25e^{-t/5}$. The half-life is $5\ln 2$ seconds.
Bacterial growth: $N' = kN$. If $N(0) = 2000$ and $N'(0) = 500$ per minute, then $k = 1/4$ and $N(t) = 2000e^{t/4}$. **10. Common mistakes**

- Integrating with respect to x while treating $y(x)$ as a constant.
- Forgetting the product rule in $y = vx$.
- Losing equilibrium solutions when dividing by a function of y.
- Dropping absolute values in logarithms without checking the domain.
- Confusing homogeneous first-order equations with homogeneous linear equations.



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